Problem 1. Let $F$ be a bounded, closed set on the line, and
$$F_\varepsilon = \bigcup_{x \in F} [x - \varepsilon, x + \varepsilon], \quad \varepsilon > 0.$$  
Prove that $\lim_{\varepsilon \to 0} mF_\varepsilon = mF$, where $m$ is the Lebesgue measure.

Problem 2. Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^\infty E_n$ has a finite Lebesgue measure, $mE < \infty$. Prove that for any set $A \subset \mathbb{R}$ (not necessarily measurable),
$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

Problem 3. Let $f \in L^1(-\infty, \infty)$. Find the limit,
$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \left( \cos x + \frac{\sin^2 x}{2} \right)^n dx,$$
and justify your answer.

Problem 4. Prove that for any integrable function $f$ on the interval $[a, b]$,
$$\lim_{n \to \infty} \int_a^b \frac{\cos(nx)}{1 + \sin^2(nx)} f(x) dx = 0.$$ 

Problem 5. Let $E \subset [0, 1]$ be a measurable set such that there exists $\varepsilon > 0$ such that
$$m(E \cap [a, b]) \geq \varepsilon |b - a|$$
for all $[a, b] \subset [0, 1]$. Prove that $mE = 1$.

Problem 6. Prove that for any function $f \in L^2[0, 1]$,
$$\|\ln(1 + |f|)\|_{L^1[0,1]} \leq \|f\|_{L^2[0,1]}.$$
Problem 1. Let \( E \) be the set of real numbers \( x \) on the interval \([0, 1]\) such that in the decimal form of \( x = 0.i_1i_2i_3 \ldots \) there is no string of four consecutive digits 2018. Prove that

1. the set \( E \) is uncountable;
2. \( mE = 0 \), where \( mE \) is the Lebesgue measure of \( E \).

Problem 2. Prove that the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}
\]
is absolutely continuous on \([0, 1]\).

Problem 3. Prove that for any integrable function \( f \) on the interval \([a, b]\),
\[
\lim_{n \to \infty} \int_{a}^{b} f(x) \sin^4(nx) \, dx = \frac{3}{8} \int_{a}^{b} f(x) \, dx.
\]

Problem 4. Let \( f(x) \in L^2(-\infty, \infty) \). Prove that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) f(x + n) \, dx = 0.
\]

Problem 5. Let \( f \in L^3[0, \pi] \) and
\[
g(x) = \frac{f(x)}{|\sin x|^{0.1}}.
\]
Prove that \( \|g\|_2 \leq 2\|f\|_3 \).

Problem 6. Let \( f(x) \) be an integrable function on \((\infty, \infty)\) and \( n \geq 1 \) an integer. Define
\[
f_n(x) = n \int_{x}^{x + \frac{1}{n}} f(t) \, dt.
\]
Prove that \( \|f_n\|_1 \leq \|f\|_1 \).
Problem 1. A real number \( x \) on \([0,1]\) is called a Liouville type number, if its decimal representation has the form

\[
x = \sum_{j=1}^{\infty} \frac{1}{10^{n_j}},
\]

where \( n_j \in \mathbb{N} \) are such that

\[
n_{j+1} - n_j \geq j!, \quad j = 1, 2, \ldots.
\]

(Liouville proved that all such numbers are transcendental.) Prove that the set of Liouville type numbers is uncountable.

Problem 2. Prove that the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{|\sin(nx)|}}
\]

converges almost everywhere on \([0, 2\pi]\).

Problem 3. Let \( f(x) \) be an absolutely continuous function on an interval \([a, b]\) such that \( f'(x) \in L^3[a, b] \). Prove that there exists a constant \( C > 0 \) such that

\[
|f(y) - f(x)| \leq C|y - x|^{2/3}
\]

for all \( a \leq x < y \leq b \).

Problem 4. Let \( f(x) \in L^1(-\infty, \infty) \). Prove that

\[
\lim_{a \to \infty} \int_{-\infty}^{\infty} \frac{f(x) \, dx}{1 + (x - a)^2} = 0.
\]

Problem 5. Let \( f \in L^6[0, 1] \), \( g \in L^7[0, 1] \), and \( h \in L^8[0, 1] \). Prove that \( fgh \in L^2[0, 1] \).

Problem 6. Let \( E_1, E_2, \ldots \) be a sequence of measurable sets on the line such that \( m(E_j \triangle E_k) < \infty \) for all \( j, k = 1, 2, \ldots \) and

\[
\lim_{j,k \to \infty} m(E_j \triangle E_k) = 0, \quad A \triangle B := (A \setminus B) \cup (B \setminus A).
\]

Prove that there exists a measurable set \( E \) such that

\[
\lim_{j \to \infty} m(E_j \triangle E) = 0.
\]
Problem 1. A real number is called transcendental if it is not a root of a polynomial with integer coefficients. Let $E$ be the set of transcendental numbers on the interval $[0, 1]$. Prove that the set $E$ is measurable and $mE = 1$.

Problem 2. Let $A$ and $E$ be disjoint sets on the line and let $E$ be measurable. Prove that

$$m^*(A \cup E) = m^*A + mE.$$ 

Problem 3. Prove that for any integrable function $f$ on the interval $[a, b]$,

$$\lim_{n \to \infty} \int_{a}^{b} f(x) |\sin(nx)| \, dx = \frac{2}{\pi} \int_{a}^{b} f(x) \, dx.$$

Problem 4. Let $f(x) \in L^1(-\infty, \infty)$. Prove that

$$\lim_{n \to -\infty} \int_{-\infty}^{\infty} f(x - n) \arctan \left( \frac{x}{n} \right) \, dx = \frac{\pi}{4} \int_{-\infty}^{\infty} f(x) \, dx.$$

Problem 5. Let $f(x)$ be a nonnegative integrable function on the interval $[a, b]$. Prove that

$$\left( \int_{a}^{b} f(x) \sin x \, dx \right)^2 + \left( \int_{a}^{b} f(x) \cos x \, dx \right)^2 \leq \left( \int_{a}^{b} f(x) \, dx \right)^2.$$

Problem 6. Let $f(x) \in L^2[0, 1]$. Define

$$g(x) = \frac{1}{x^{4/3}} \int_{0}^{x} f(t) \, dt.$$

Prove that $\|g\|_1 \leq 6\|f\|_2$. 

Problem 1. Let \( A \) and \( B \) be two sets on the interval \([0, 1]\) such that
\[
\overline{A} \cap \overline{B} = \emptyset,
\]
where \( \overline{A} \) and \( \overline{B} \) are the closures of the sets \( A \) and \( B \), respectively, and \( \emptyset \) is the empty set. Prove that
\[
m^*(A \cup B) = m^*A + m^*B,
\]
where \( m^* \) is the outer measure.

Problem 2. Let \( f(x) \) be an absolutely continuous function on \([0, 1]\) and \( E \) a set on \([0,1]\) such that \( mE = 0 \), where \( mE \) is the Lebesgue measure of the set \( E \). Let \( f(E) \) be the image of the set \( E \) with respect to \( f \),
\[
f(E) = \{ y \in \mathbb{R} \mid \exists x \in [0, 1] \text{ such that } f(x) = y \}.
\]
Prove that \( mf(E) = 0 \).

Problem 3. Consider the set \( E \) of numbers \( x \) on \([0, 1]\) such that in every 100 consecutive digits \( i_{k+1}, \ldots, i_{k+100}, k = 0, 1, 2, \ldots, \) of a decimal fraction representing \( x \), \( x = 0.i_1i_2i_3\ldots \), where \( i_k \in \{0,1,\ldots,9\} \), there is at least one 0. Prove that \( mE = 0 \).

Problem 4. Let \( f(x) \) be an integrable function on the interval \([a, b]\). Define recursively the functions,
\[
f_0(x) = f(x), \quad f_k(x) = \int_a^x f_{k-1}(t) \, dt, \quad k \geq 1; \quad a \leq x \leq b.
\]
Prove that the series
\[
F(x) = \sum_{k=1}^{\infty} f_k(x)
\]
converges uniformly on \([a, b]\) and \( F(x) \) is an absolutely continuous function.

Problem 5. Let \( A \) be a measurable subset of \([0, 2\pi]\). Prove that
\[
\lim_{n \to \infty} \int_A \cos^2(nx) \, dx = \frac{1}{2} mA,
\]
where \( mA \) is the measure of \( A \).

Problem 6. Let \( f(x) \) be an integrable function on \([0, \infty)\). Prove that
\[
\lim_{n \to \infty} n \int_0^1 x f(nx) \, dx = 0.
\]
Problem 1: Show that there is a compact set $K \subset [0, 1]$ with Lebesgue measure $m(K) > 0$, but such that $K$ contains no non-empty open interval.

Problem 2: Suppose $(E_n)$ is a sequence of measurable subsets of $[0, 1]$ with Lebesgue measure $m(E_n \Delta E_{n+1}) \leq 1/2^n$ for all $n \geq 1$. Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

a) Show that $(1_{E_n})$ is a Cauchy sequence in $L_1$-norm in $L_1[0, 1]$.

b) Show that there is a measurable set $E$ such that $m(E \Delta E_n) \to 0$ as $n \to \infty$.

Problem 3: Let $f \geq 0$ be a measurable function on $[0, 1]$. Show that if $\sum_{n=1}^{\infty} m\{f \geq n^2\} < \infty$, then $\int_0^1 \sqrt{f(x)}\, dm(x) < \infty$.

Problem 4: Describe a measurable function $f$ on $[0, 1]$ such that $f \in L_1[0, 1]$, but for all $1 < p \leq \infty$, $f \notin L_p[0, 1]$.

Problem 5: Suppose $h : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^1 h(x)\, dx = 0$. For $x \geq 0$, let $\{x\}$ be the fractional part of $x$: $\{x\} = x - p$ where $p$ is a positive integer such that $p \leq x < p + 1$.

a) Show that for any $0 \leq a \leq b \leq 1$, we have $\int_a^b h(\{nx\})\, dx \to 0$ as $n \to \infty$.

b) Using a), show that for any $g \in L_1[0, 1]$, we have $\int_0^1 h(\{nx\}) g(x)\, dx \to 0$ as $n \to \infty$.

Problem 6: Show that for $f \in L_1(\mathbb{R})$ and any $\epsilon > 0$, we have

$$\lim_{n \to \infty} m\{x \in \mathbb{R} : |f(x + \frac{1}{n}) - f(x)| \geq \epsilon\} = 0.$$
Throughout this examination, $m$ denotes the Lebesgue measure on $\mathbb{R}$.

**Problem 1:** Show that there is a compact set $K \subset [0, 1]$ with $m(K) \geq \frac{11}{12}$, but such that $K$ contains no non-empty open intervals.

**Problem 2:** Suppose $(f_n)$ is a sequence in $L_1[0, 1]$ such that $\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dm(x) = 0$. Show that for all $\delta > 0$, we have $m\{x \in [0, 1] : |f_n(x)| \geq \delta\} \to 0$ as $n \to \infty$.

**Problem 3:** Let $f \geq 0$ be a measurable function on $[0, 1]$. Assume $m\{f \geq x\} \leq \frac{1}{x^2}$ for all $x \geq 1$. Show that $f \in L_r[0, 1]$ for all $1 \leq r < 2$.

**Problem 4:** Show that for $p, 1 < p \leq \infty$, $L_p[0, 1] \subset L_1[0, 1]$ but $L_p(\mathbb{R})$ is not a subset of $L_1(\mathbb{R})$.

**Problem 5:** Let $(r_n)$ be the sequence of functions such that $r_n : [0, 1) \to \{-1, 1\}$ given by the rule that $r_n(x) = (-1)^k$ for $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ with $k = 0, \ldots, 2^n - 1$. Let $f \in L_1[0, 1]$. Show that $\lim_{n \to \infty} \int_0^1 f(x)r_n(x) \, dm(x) = 0$.

**Problem 6:** Show that for $f \in L_1(\mathbb{R})$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f(x + 1/n) - f(x)| \, dm(x) = 0.$$
Problem 1. Let \( E \subset [0, 1] \) be a measurable set such that there exists \( \varepsilon > 0 \) such that
\[
m(E \cap [a, b]) \geq \varepsilon |b - a|
\]
for all \([a, b] \subset [0, 1]\). Prove that \( mE = 1 \).

Problem 2. Let \( E \) be a measurable set on the line and \( f(x) \) a nonnegative measurable function on \( E \) such that the limit,
\[
L = \lim_{n \to \infty} \int_E |f(x)|^n dx,
\]
exists, where \( 0 < L < \infty \). Prove that \( m\{x \in E \mid f(x) = 1\} = L \).

Problem 3. Let \( f \) be an integrable function on a set \( E \) of a finite Lebesgue measure \( mE < \infty \). Let \( E_n = \{x \in E \mid |f(x)| \geq n\} \). Prove that
\[
sn \sum_{n=0}^{\infty} mE_n \geq \int_E |f(x)| \, dx \geq \sum_{n=1}^{\infty} mE_n.
\]

Problem 4. Prove that if \( f \) is absolutely continuous on \([a, b]\) and \( g(x) = e^{|f(x)|} \) then \( g \) is absolutely continuous on \([a, b]\) as well.

Problem 5. Prove that if \( f \in L^1[0, \infty) \), then
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^n x f(x) \, dx = 0.
\]

Problem 6. Prove that if \( f(x) \in L^{4+\varepsilon}[0, 1] \), where \( \varepsilon > 0 \), and \( g(x) = f(x^2) \), then \( g(x) \in L^2[0, 1] \).
Problem 1. Let \( f \) be a measurable function on \([0, 1]\) such that \( f \) is finite almost everywhere. Prove that for any \( \varepsilon > 0 \), there is a polynomial \( P(x) = \sum_{k=0}^{n} p_k x^k \), such that
\[
m \{ x \mid |f(x) - P(x)| \geq \varepsilon \} \leq \varepsilon,
\]
where \( m \) is the Lebesgue measure.

Problem 2. Let \( \mathcal{M} \) be the \( \sigma \)-algebra of all Lebesgue measurable sets on \([0, 1]\). Prove that there is a countable collection of sets \( \mathcal{D} \subset \mathcal{M} \) such that for every set \( E \in \mathcal{M} \) and every \( \varepsilon > 0 \) there is a set \( A \in \mathcal{D} \) such that
\[
m(A \Delta E) \leq \varepsilon, \quad A \Delta E \equiv (A \setminus E) \cup (E \setminus A).
\]

Problem 3. Let \( f \) be a measurable function on \([0, 1]\) such that \( 0 \leq f(x) \leq 1 \) for all \( x \) on \([0, 1]\), and let \( g \) be a function of bounded variation on \([0, 1]\). Prove that the function \( g(f(x)) \) is measurable on \([0, 1]\).

Problem 4. Let \( f \) be a continuous function on \([0, 1]\), which is differentiable almost everywhere. Suppose that \( f' \in L^p[0, 1] \), where \( p > 1 \). Let \( \alpha = 1 - \frac{1}{p} \). Prove that there is a constant \( C > 0 \) such that
\[
|f(y) - f(x)| \leq C|y - x|^{\alpha},
\]
for all \( x, y \) on \([0, 1]\).

Problem 5. Let \( f \in L^\infty[0, 1] \). Prove that for every \( p \geq 1 \),
\[
\exp \left[ \int_0^1 f(x) \, dx \right] \leq \|e^{f(x)}\|_p.
\]

Problem 6. Prove that for every \( p < 1 \) the function
\[
f(x) = |\sin (x^{-1})|^{-p}
\]
is integrable on \([0, 1]\).
Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that
\[ m(A \cap B) = 0.3, \quad m(A \cap C) = 0.4, \quad m(B \cap C) = 0.2, \]
\[ m(A \cup B) = 0.8, \quad m(A \cup C) = 0.8, \quad m(B \cup C) = 0.9, \]
\[ m(A \cup B \cup C) = 1. \]
where $m$ is the Lebesgue measure. Find $m(A \cap B \cap C)$.

Problem 2. For real numbers $a, b, c$, let $s[a, b, c]$ denote the second largest number among $a, b, c$. For example, $s[1, 10, 3] = 3$, $s[5, 3, 5] = 5$, and $s[2, 2, 2] = 2$. Let $f_1(x), f_2(x), f_3(x)$ be three measurable functions on a set $E$ on the real line. Prove that the function $g(x) = s[f_1(x), f_2(x), f_3(x)]$ is measurable as well.

Problem 3. Let $F$ be a bounded, closed set on the line, and
\[ O_n = \bigcup_{x \in F} \left( x - \frac{1}{n}, x + \frac{1}{n} \right). \]
Prove that
\[ \lim_{n \to \infty} mO_n = mF, \]
where $m$ is the Lebesgue measure.

Problem 4. Prove that there is a countable collection \( \{E_1, E_2, \ldots\} \) of measurable sets on $[0, 1]$ such that for any measurable set $E$ on $[0, 1]$ and any $\varepsilon > 0$ there exists a set $E_n$ from the collection such that
\[ m(E \Delta E_n) \leq \varepsilon, \]
where $m$ is the Lebesgue measure and $E \Delta E_n = (E \setminus E_n) \cup (E_n \setminus E)$.

Problem 5. Let us enumerate all rational points on $[0,1]$, \( \mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots\} \).
Consider the function
\[ f(x) = \sum_{n=1}^{\infty} \frac{\theta(x - r_n)}{n^{3/2}} \]
on $[0,1]$, where $\theta(x)$ is the Heaviside step function, $\theta(x) = \chi_{[0,\infty)}(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$
Prove that $f'(x) = 0$ almost everywhere.

Problem 6. Prove that if $f_n$ converges to $f$ in $L^4[E]$ and $g_n$ converges to $g$ in $L^6[E]$, where $mE < \infty$, then $f_ng_n$ converges to $fg$ in $L^2[E]$. 
Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that 
\[ m(A \Delta B) + m(B \Delta C) = m(A \Delta C), \]
where $m$ is the Lebesgue measure and as usual, 
\[ A \Delta B = (A \setminus B) \cup (B \setminus A). \]
Prove that 
\[ m[B \setminus (A \cup C)] = 0. \]

Problem 2. Let $E$ be a measurable set on the line and $f$ a function on $E$ such that the set 
\[ \{x \in E \mid f(x) < a \sin x\} \]
is measurable for any real $a$. Prove that the function $f$ is measurable.

Problem 3. Let us enumerate all rational numbers on $[0, 1]$, \( \mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots \}. \)
Prove that the function 
\[ f(x) = \sum_{k=1}^{\infty} \frac{|x - r_k|^{1/2}}{k^{3/2}} \]
is differentiable almost everywhere on $[0, 1]$.

Problem 4. Let $f \in L^2[0, 1]$. Prove that 
\[ \left[ \int_0^1 |f(x) - \int_0^1 f(x) dx| dx \right]^2 \leq \int_0^1 f^2(x) dx - \left( \int_0^1 f(x) dx \right)^2. \]

Problem 5. Let $T(f)$ be a bounded linear functional on $L^2[0, 1]$ such that \( T(x^n) = \frac{1}{n+1} \) for $n = 0, 1, 2, \ldots$. Prove that 
\[ T(f) = \int_{[0,1]} f(x) dx \]
for all $f \in L^2[0, 1]$.

Problem 6. Prove that the sequence of functions, 
\[ f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^{3/6}}, \quad n = 1, 2, \ldots, \]
is a Cauchy sequence in $L^2[0, 2\pi]$. 

QUALIFYING EXAM IN REAL ANALYSIS
IUPUI, AUGUST 2014
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Problem 1. Let $A, B$ and $C$ be measurable sets on the line such that
\[ m(A \cap B) = 0.9, \quad m(A \cap C) = 0.9, \quad m(B \cap C) = 0.9, \quad m(A \cup B \cup C) = 1. \]
Prove that
\[ m(A \cap B \cap C) \geq 0.85. \]

Problem 2. Suppose that a sequence $E_1, E_2, \ldots$ of measurable sets on $[0, 1]$ satisfies the Cauchy condition, so that for every $\varepsilon > 0$ there exists $N > 0$ such that
\[ m(E_n \Delta E_m) \leq \varepsilon, \]
for all $n, m \geq N$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Prove that there exists a measurable set $E$ on $[0, 1]$ such that
\[ \lim_{n \to \infty} m(E_n \Delta E) = 0. \]

Problem 3. Evaluate the limit,
\[ \lim_{n \to \infty} \int_0^1 \frac{nx^n dx}{(1 + x^n)^2}, \]
and justify your answer.

Problem 4. Let us enumerate all rational points on $[0, 1]$,
\[ \mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}. \]
Consider the function
\[ f(x) = \sum_{n=1}^{\infty} \frac{|x - r_n|}{n^{3/2}} \]
on $[0, 1]$. Prove that the function $f(x)$ is differentiable almost everywhere.

Problem 5. Prove that if $f_n$ converges to $f$ in $L^p(E)$ and $g_n$ converges to $g$ in $L^q(E)$, where
\[ \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad p > 0, \quad q > 0, \quad r \geq 1, \]
then $f_ng_n$ converges to $fg$ in $L^r(E)$.

Problem 6. Let
\[ f_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k^{2/3}}. \]
Prove that there exists an integrable function $f(x)$ on the interval $[0, 2\pi]$ such that
\[ \lim_{n \to \infty} \int_{[0,2\pi]} |f(x) - f_n(x)| dx = 0. \]
Problem 1. Let $E$ be a set on the line of a finite outer measure, $m^*E < \infty$, and 
\[ S = \sup_{\text{measurable } A, A \subseteq E} m A. \]
Prove that $S = m^*E$ if and only if $E$ is measurable.

Problem 2. Prove that if $f(x)$ is an absolutely continuous function on the segment $[0, 1]$ such that $f'(x) = 1$ almost everywhere, then $f(x) = x + C$, where $C$ is a constant.

Problem 3. Let \{f_n(x), n = 1, 2, \ldots\} be a sequence of integrable functions on the segment $[0, 1]$ such that 
\[ \sum_{n=1}^{\infty} \|f_n\|_{L^1[0,1]} < \infty. \]
Prove that the series 
\[ \sum_{n=1}^{\infty} \sin f_n(x) \]
converges in $L^1[0,1]$.

Problem 4. Let $F(f)$ be a bounded linear functional on $L^1[0,1]$ such that $F(f) = 0$ for all $f$ such that 
\[ \int_0^1 f(x)dx = 0. \]
Prove that then there is a real number $C$ such that for all $f \in L^1[0,1]$, 
\[ F(f) = C \int_0^1 f(x)dx. \]

Problem 5. Prove that if $f_n$ converges to $f$ in $L^3[0,1]$, then $(f_n)^2$ converges to $f^2$ in $L^1[0,1]$.

Problem 6. Let $f$ be an integrable function on the segment $[0, 1]$ such that for any $0 \leq a < b \leq 1$, 
\[ \left| \int_a^b f(x)dx \right| \leq (b - a)^2. \]
Prove that $f = 0$ almost everywhere.
Problem 1. Prove that if \( A \) is a set on \([0, 1]\) and \( A_n \) its \((1/n)\)-neighborhood, i.e.,
\[
A_n = \left\{ x \in \mathbb{R} : \exists y \in A, \quad |x - y| < \frac{1}{n} \right\},
\]
then
\[
\bigcap_{n=1}^{\infty} A_n = \overline{A},
\]
where \( \overline{A} \) is the closure of \( A \).

Problem 2. Prove that the collection of all measurable sets on \([0, 1]\) is separable, so that there exists a countable collection \( \mathcal{M}_0 = \{E_1, E_2, \ldots\} \) of measurable sets on \([0, 1]\) such that for any measurable set \( E \) on \([0, 1]\) and any \( \varepsilon > 0 \) there exists a set \( E_n \in \mathcal{M}_0 \) such that
\[
m(E \triangle E_n) \leq \varepsilon,
\]
where \( E \triangle E_n = (E \setminus E_n) \cup (E_n \setminus E) \).

Problem 3. Let us enumerate all rational points on \([0, 1]\),
\[
\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.
\]
Consider the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 |x - r_n|^{1/2}}
\]
on \([0,1]\). Prove that the series defining the function \( f(x) \) converges for almost all \( x \) and \( f \) is integrable.

Problem 4. Prove that if a sequence \( \langle f_n \rangle \) of integrable functions on a measurable set \( E \) converges to a function \( f \) in such a way that
\[
\int_E |f_n(x) - f(x)| \, dx \leq \frac{1}{2^n}, \quad n = 1, 2, \ldots,
\]
then \( f_n(x) \) converges to \( f(x) \) a.e. on \( E \).

Problem 5. Prove that if \( f \in L^2[0, 1] \) and
\[
\int_0^1 f(x) \, dx = 0,
\]
then
\[
\int_0^1 |f(x) - x|^2 \, dx \geq \frac{1}{4}.
\]
Problem 6. Let $f(x)$ be an absolutely continuous function on an interval $[a, b]$ such that $f'(x) \in L^2[a, b]$. Prove that there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^{1/2}$$

for all $x, y \in [a, b]$. 
Problem 1. Let $f$ be a continuously differentiable function on $[0, 2\pi]$ such that $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$. Prove that for any $\varepsilon > 0$, there is a trigonometric polynomial,

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \sin(kx) + b_k \cos(kx)],$$

such that

$$\sup_{0 \leq x \leq 2\pi} |f(x) - g(x)| + \sup_{0 \leq x \leq 2\pi} |f'(x) - g'(x)| \leq \varepsilon.$$

Problem 2. Prove that for any Lebesgue measurable set $E$ on $[0, 1]$ and any $\varepsilon > 0$, there is a finite collection of disjoint intervals $(a_j, b_j), 1 \leq j \leq n,$ on $[0, 1]$ such that all the end-points \{a_j, b_j\} are rational and

$$m \left( E \triangle \bigcup_{j=1}^{n} (a_j, b_j) \right) \leq \varepsilon,$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Problem 3. Let $f$ satisfy a Lipschitz condition on $[0, 1]$, so that there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x$ and $y$ on $[0, 1]$. Prove that $f$ is differentiable almost everywhere on $[0, 1]$.

Problem 4. Let $f \in L^2[0, 1]$. Prove that

$$\inf_{-\infty < a < \infty} \int_{0}^{1} |f(x) - a| dx \leq \sqrt{\int_{0}^{1} f^2(x) dx - \left( \int_{0}^{1} f(x) dx \right)^2}.$$

Problem 5. The Sierpinski carpet is a set on the plane which is constructed as follows. We begin with a square. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the open central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. Prove that the planar Lebesgue measure of the Sierpinski carpet is equal to 0.

Problem 6. Prove that for any $\infty > p \geq 1$ there is a constant $C_p > 0$ such that for any integrable function $f$ on $[0, 1]$,

$$\| \ln(1 + |f|) \|_{L^p[0,1]} \leq C_p \left( 1 + \|f\|_{L^1[0,1]} \right).$$
1. Assume that $B_n \subset A_n \subset \mathbb{R}$ for $n = 1, 2, \ldots$ are measurable sets. Prove that
$$m\left(\bigcup_{n=1}^{\infty} A_n \setminus \bigcup_{n=1}^{\infty} B_n\right) \leq \sum_{n=1}^{\infty} (m(A_n) - m(B_n)).$$

2. Let $A_1, A_2, A_3, A_4$ be measurable subsets of $[0, 1]$, such that
$$\sum_{k=1}^{4} m(A_k) > 3.$$ Prove that
$$m\left(\bigcap_{k=1}^{4} A_k\right) > 0.$$ 

3. Consider a sequence of functions $f_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, convergent to $f$ in $L^1(\mathbb{R})$ and to $g$ in $L^2(\mathbb{R})$. Prove that $f = g$ a.e.

4. Let $E$ be a measurable subset of $\mathbb{R}$ and $f \in L^1(E)$. Set $A_n = \{x \in E : |f(x)| \geq n\}$ for $n = 1, 2, \ldots$. Prove that
$$\sum_{n=1}^{\infty} m(A_n) \leq \int_{E} |f|.$$ 

5. Let $p, q, r \in (1, \infty)$ with $1/p + 1/q + 1/r = 1$. Prove that for every functions $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ and $h \in L^r(\mathbb{R})$
$$\int_{\mathbb{R}} |fgh| \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r.$$ 

6. Let $f \in L^1([0, 1])$. Prove that $\sqrt{1 + f^2} \in L^1([0, 1])$ and
$$\sqrt{1 + (\|f\|_1)^2} \leq \left\|\sqrt{1 + f^2}\right\|_1.$$ 

Hint: For inequality, use Jensen’s inequality.
1. Prove that for every $\varepsilon > 0$ there exists an open set $A \subset \mathbb{R}$, dense in $\mathbb{R}$, of Lebesgue measure less than $\varepsilon$.

2. Prove that if a function $f : [0,1] \to \mathbb{R}$ has bounded variation and $\inf \{|f(x)| : x \in [0,1]\} > 0$, then $1/f$ also has bounded variation.

3. Let $a < b$ be real numbers. Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function with bounded derivative and such that $f(a) = 0$. Prove that

$$\sup\{|f(x)| : x \in [a, b]\} \leq \sqrt{(b-a) \int_a^b (f'(x))^2 \, dx}.$$ 

4. Give example of a function $f : [0,1] \to \mathbb{R}$ which is integrable, but is not essentially bounded on any subinterval of $[0,1]$.

5. Show that $f \mapsto \sqrt{\|f\|_1 \cdot \|f\|_\infty}$ is not a norm in $L^\infty([0,1])$.

6. For $n = 1, 2, 3, \ldots$ define a function $f_n : [0,1) \to \mathbb{R}$ as follows. Let $k(x, n) = \lfloor 2^n x \rfloor$ (the largest integer smaller than or equal to $2^n x$); let

$$f_n(x) = \begin{cases} 
0 & \text{if } k(x, n) \text{ is even}, \\
1 & \text{if } k(x, n) \text{ is odd}.
\end{cases}$$

Does there exist a subsequence of the sequence $(f_n)_{n=1}^\infty$ convergent almost everywhere? Explain.
IUPUI QUALIFYING EXAM ON REAL ANALYSIS
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PAVEL BLEHER

Problem 1. Consider the space \( l \) of sequences \( x = (x_1, x_2, \ldots) \) such that the limit, \( \lim_{n \to \infty} x_n \), exists. Define the norm in \( l \) as the \( l^\infty \)-norm,

\[ \|x\|_\infty = \sup_n |x_n|. \]

Is the space \( l \) complete or not? Explain.

Problem 2. Prove that the linear functional

\[ F[f] = \int_0^1 xf(x) \, dx \]

on \( L^3[0, 1] \) is bounded and evaluate its norm.

Problem 3. Let \( f \in L^1(-\infty, \infty) \). Find the limit,

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \left( \cos x + \frac{\sin^3 x}{2} \right)^n \, dx, \]

and justify your answer.

Problem 4. Let \( C \) be the Cantor ternary set,

\[ C = [0, 1] \setminus \left[ \left( \frac{1}{3}, \frac{2}{3} \right) \cup \left( \frac{1}{9}, \frac{8}{9} \right) \cup \left( \frac{7}{9}, \frac{8}{9} \right) \cup \ldots \right], \]

and let

\[ E = \{ x + y \mid x \in C, \; y \in \mathbb{Q} \}, \]

where \( \mathbb{Q} \) is the set of rational numbers. Find the Lebesgue measure of \( E \) and justify your answer.

Problem 5. Prove that if \( f \in L^1(-\infty, \infty) \) and

\[ g(x) = \int_{-\infty}^{\infty} f(x - y)e^{-|y|} \, dy, \]

then \( g \in L^1(-\infty, \infty) \) and

\[ \|g\|_1 \leq 2\|f\|_1. \]

Problem 6. Let \( (X, \mathcal{A}, \mu) \) be a measure space. Prove that if \( f \in L^6[X] \) and \( g \in L^3[X] \), then \( fg \in L^2[X] \).
Problem 1. The Riemann function $R(x)$ on the segment $[0,1]$ is defined as

$$R(x) = \begin{cases} 
0, & \text{when } x \text{ is irrational or } x = 0, \\
\frac{1}{q}, & \text{when } x = \frac{p}{q} > 0, \text{ where } p \text{ and } q \text{ are relatively prime.}
\end{cases}$$

Is there a sequence $\langle f_n \rangle$ of continuous functions on $[0,1]$ which converges to $R(x)$ at every point $x \in [0,1]$? Explain.

Problem 2. Let us enumerate all rational points on the segment $[0,1]$, 
$$\mathbb{Q} \cap [0,1] = \{ r_1, r_2, \ldots \}.$$ Define the function

$$f(x) = \sum_{n: \ r_n < x} \frac{1}{2^n}.$$ 

Prove that

1. the function $f(x)$ is discontinuous at every rational point,
2. the function $f(x)$ is continuous at every irrational point,
3. the function $f(x)$ is differentiable almost everywhere,
4. $f'(x) = 0$ almost everywhere.

Problem 3. Let $E_1 \subset E_2 \subset \ldots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^{\infty} E_n$ has a finite Lebesgue measure. Prove that for any set $A \subset \mathbb{R}$,

$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

Problem 4. Let $f \in L^1(-\infty, \infty)$. Find the limit,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{f(x)e^{nx}dx}{1 + e^{nx}},$$

and justify your answer.

Problem 5. Consider the functional

$$F(f) = \int_0^1 e^x f(x)dx.$$ 

Prove that $F$ is a linear bounded functional on $L^4[0,1]$ and calculate the norm of $F$ on $L^4[0,1]$. 
Problem 6. Prove that if \( f, g \in L^1(-\infty, \infty) \) and
\[
h(x) = \int_{-\infty}^{\infty} f(x - e^y)g(y)dy,
\]
then \( h \in L^1(-\infty, \infty) \) and
\[
\|h\|_1 \leq \|f\|_1 \|g\|_1.
\]
Problem 1. Does there exist a measurable set of non-zero measure which is closed and has no interior points?

Problem 2. Let \(\{f_n\}\) be a sequence of measurable functions on \((0, 1)\). Show that the set
\[
E = \{x \in (0, 1) \mid \{f_n(x)\} \text{ is a convergent sequence}\}
\]
is measurable.

Problem 3. Enumerate all rational points on \([0, 1]\), so that
\[
\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \ldots\}.
\]
Define the function
\[
f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} \theta(x - r_n), \quad 0 \leq x \leq 1,
\]
where
\[
\theta(x) = \begin{cases} 
1, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]
Prove that there exists a point \(0 < x_0 < 1\) such that \(f(x)\) is differentiable at \(x_0\).

Problem 4. Let \(g(x)\) be a continuous periodic function of period 1 on \(\mathbb{R}\). Prove that for any integrable function \(f(x)\) on \([0, 1]\),
\[
\lim_{n \to \infty} \int_0^1 f(x)g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx
\]
Problem 5. Prove that the operator
\[
A : f(x) \to \int_{-\infty}^{+\infty} f(x-y)e^{-y^2} \, dy
\]
is a bounded operator in \(L^2(-\infty, \infty)\).

Problem 6. Let \((X, d)\) be a compact metric space. Let \(\{f_n\}\) be a sequence of continuous real valued functions on \(X\) such that \(f_n(x) \leq f_{n+1}(x)\) for all \(n, x\). Assume that the sequence \(\{f_n\}\) is pointwise convergent and let
\[
f(x) = \lim_{n \to \infty} f_n(x).
\]
Show that \(f(x)\) is continuous if and only if the sequence \(\{f_n\}\) converges uniformly.
Problem 1. Prove that the space $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$, with the norm,
\[
\|f(x)\|_{C^1[0, 1]} = \sup_{0 \leq x \leq 1} (|f(x)| + |f'(x)|),
\]
is separable.

Problem 2. Find a linear function $ax + b$ such that the norm,
\[
\|ax + b - e^x\|_{L^2[0, 1]},
\]
is minimal.

Problem 3. Let $f(x)$ be a continuous function on $\mathbb{R}$, and let $g(x)$ be a measurable function on $[0, 1]$. Is the function $f(g(x))$ measurable? Explain.

Problem 4. Is there an integrable function $f(x)$ on $[0, 1]$ such that for any $0 < a < b < 1$,
\[
\|f(x)\|_{L^2[a,b]} = \infty.
\]
Explain.

Problem 5. Is there an open connected set in $\mathbb{R}^2$ of finite Lebesgue measure, which has a non-empty intersection with any straight line? Explain.

Problem 6. Prove that there is a constant $C > 0$ such that for any continuous function $f$ on $[0, 1]$,
\[
\|\ln(1 + |f|)\|_{L^4[0, 1]} \leq C\|f\|_{L^2[0, 1]}.
\]
Problem 1. Prove that any set $X$ on the line is separable, i.e., there exists an at most countable set $A \subset X$ whose closure contains $X$.

Problem 2. Is there a measurable set $E$ on $[0, 1]$ such that for any $0 \leq a < b \leq 1$, 
$$m(E \cap [a, b]) = \frac{b - a}{2},$$
where $m$ stands for the Lebesgue measure?

Problem 3. Prove that if $f$ is integrable on $[0, 1]$, then
$$\lim_{n \to \infty} n \int_0^1 \sin(x/n) f(x) \, dx = \int_0^1 x f(x) \, dx.$$

Problem 4. Prove that if functions $f_1$, $f_2$ are absolutely continuous on $[0, 1]$, then the function $f(x) = \max\{f_1(x), f_2(x)\}$ is absolutely continuous as well.

Problem 5. Let $f_1, f_2, \ldots$ be a sequence of nondecreasing absolutely continuous functions on the closed interval $[0, 1]$ such that the series 
$$f(x) = \sum_{n=1}^\infty f_n(x)$$
converges at every point $x \in [0, 1]$. Prove that the function $f$ is absolutely continuous.

$Hint$. Consider the series of the derivatives,
$$\sum_{n=1}^\infty f'_n(x),$$
and prove that it converges a.e. to an integrable function.

Problem 6. Suppose that $f \in L^3[0, 1]$. Prove that
$$\|xf(x)\|_2 \leq \|f(x)\|_3^{7/6}.$$

$Hint$. Use the Hölder inequality.
Problem 1. Suppose that $A \subset \mathbb{R}$ is a measurable set. Prove that the set
$$B = A \cup (-A), \quad -A \equiv \{x : -x \in A\},$$
is measurable and
$$mA \leq mB \leq 2mA,$$
where $m$ is the Lebesgue measure.

Problem 2. Prove that for any integrable function $f$ on $[a, b]$,
$$\lim_{n \to \infty} \int_a^b f(x) \sin^3(nx) \, dx = 0.$$  

Problem 3. Prove that there is no function $f(x) \in L^\infty[0, 1]$ such that
$$\lim_{n \to \infty} \int_0^1 [f(x)]^n \, dx = 2.$$  

Problem 4. For a given $\varepsilon > 0$, construct an open dense set $O$ on $[0, 1]$ such that $mO < \varepsilon$, where $m$ is the Lebesgue measure.

Problem 5. Prove that if $f \in L^2[0, 1]$ and $g \in L^3[0, 1]$ then $fg \in L^{6/5}[0, 1]$.

Problem 6. Prove the existence of the limit,
$$\lim_{n \to \infty} \int_0^\infty \frac{1 + x^n}{2 + x^n} e^{-x} \, dx,$$
and find it.
Problem 1. Let $F$ be a closed subset of $[0,1]$ and $F_{\varepsilon}$, where $\varepsilon > 0$, its $\varepsilon$-neighborhood, i.e.,

$$F_{\varepsilon} = \{ x \in \mathbb{R} : \exists y \in F, \ |x - y| < \varepsilon \}.$$ 

Prove that

$$\lim_{\varepsilon \to 0} mF_{\varepsilon} = mF,$$

where $m$ is the Lebesgue measure. Does this statement hold in the case when $F$ is an arbitrary open subset of $[0,1]$? Explain.

Problem 2. Let $A$ and $B$ be two bounded closed subsets of $\mathbb{R}$. Prove that the set

$$A + B = \{ x + y : x \in A, y \in B \}.$$

is closed and

$$m(A + B) \geq mA + mB,$$

where $m$ is the Lebesgue measure.

Problem 3. Let $f_n : \mathbb{R} \to \mathbb{R}^+$ be a sequence of nonnegative integrable functions on the line such that

$$\lim_{n \to \infty} f_n(x) = 0$$

almost everywhere. Prove that

$$\lim_{n \to \infty} \int_{\mathbb{R}} \min_{1 \leq k \leq n} f_k(x) \, dx = 0.$$ 

Problem 4. Let $f$ be a nonnegative integrable function on the interval $[0,1]$ such that

$$\int_0^1 f(x) \, dx = 1,$$

and let $g(x) = \sqrt{f(1-x)}$. Prove that the function $g$ is integrable on $[0,1]$ and

$$\int_0^1 g(x) \, dx \leq 1.$$ 

Problem 5. Let $f$ be an integrable function on $[0,1]$ such that for any $0 \leq a < b \leq 1$,

$$\int_a^{(a+b)/2} f(x) \, dx = \int_{(a+b)/2}^b f(x) \, dx.$$ 

Prove that then the function $f$ is constant almost everywhere.
Problem 6. Let $f$ be a real valued function on the interval $(0, 1)$. A point $x$ is called a point of discontinuity of $f$ of the first kind, if the limits of $f$ at $x$ both from the right and from the left exist and they are not equal. Prove that the set of points of discontinuity of $f$ of the first kind is at most countable.

Hint. Prove first that the set of points of discontinuity of $f$ of the first kind, with the difference of the right and the left limits bigger than $1/n$, is at most countable for any $n = 1, 2, \ldots$. 
Problem 1. Suppose that $A \subset [0, 1]$ is a measurable set. Prove that the set
$$B = \cos A = \{ \cos x, \ x \in A \}$$
is measurable and $mB \leq 0.85mA$, where $m$ is the Lebesgue measure.

_Hint._ Use that $\sin 1 = 0.841 \ldots$

Problem 2. Prove that for any integrable function $f$ on $[a, b]$,
$$\lim_{n \to \infty} \int_a^b f(x) \sin^2(nx) dx = \frac{1}{2} \int_a^b f(x) dx.$$

Problem 3. Prove that there is no positive integrable function $f(x)$ on $[0, 1]$ such that
$$\lim_{n \to \infty} \int_0^1 [f(x)]^n dx = 2.$$

Problem 4. Let $A$ be a measurable set on $[0, 1]$. A point $x \in [0, 1]$ is called a _density point_ of $A$ if
$$\lim_{\varepsilon \to 0} \frac{m(A \cap [x, x + \varepsilon])}{|\varepsilon|} = 1,$$
where $m$ stands for the Lebesgue measure. Prove that almost all points of the set $A$ are its density points.

_Hint._ Apply the theorem on the differentiation of an integral.

Problem 5. Let $f \in L^p(X)$, where $(X, \mathcal{A}, \mu)$ is a finite measure space, $\mu(X) < \infty$, and let $p > r \geq 1$. Prove that then $f \in L^r(X)$ and
$$\|f\|_r \leq (\mu X)^{\frac{1}{r} - \frac{1}{p}} \|f\|_p.$$

Problem 6. Prove that there exists the limit,
$$\lim_{n \to \infty} \int_0^1 \frac{e^{-x} \cos x}{nx^2 + \frac{1}{n}} dx,$$
and find it.

_Hint._ Make a change of variable.