Math 554 Qualifying Exam

January, 2019

You may use any theorems from the textbook. Any other claims must be proved in details.

1. Let $F$ be a field and $m$ and $n$ be positive integers. Prove the following.
   (a) If $A$ and $B$ are $m \times n$ matrices over $F$, then $\text{rank}(A + B) \leq \text{rank}A + \text{rank}B$;
   (b) If $A$ and $B$ are $n \times n$ matrices over $F$ such that $AB = 0$, then $\text{rank}A + \text{rank}B \leq n$;

2. Let $S$ and $T$ be linear transformations on an $n$-dimensional vector space over $\mathbb{C}$. Show that if $ST = TS$, then $S$ and $T$ have a common eigenvector.

3. Let $v \in \mathbb{R}^n$ be a nonzero column vector. Find an orthogonal matrix $Q$ and a diagonal matrix $D$, such that $vv^T = QDQ^T$. (Hint: Find the eigenvalues without computing the characteristic polynomial!)

4. Let $A$ be a $2 \times 2$ real valued matrix with 2 distinct complex eigenvalues $a \pm bi$ with $b \neq 0$. Show that $A$ is similar over $\mathbb{R}$ to the matrix $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

5. Show that if $A$ is an invertible $3 \times 3$ matrix over $\mathbb{C}$, then there is a $3 \times 3$ matrix $B$ over $\mathbb{C}$ such that $B^3 = A$. (Hint: Recall the binomial series of $(1+x)^{1/3} = 1+C(\frac{1}{3}, 1)x+C(\frac{1}{3}, 2)x^2+C(\frac{1}{3}, 3)x^3+\cdots = 1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\cdots$)

6. Suppose $T$ is a linear transformation on a finite dimensional vector space $V$ over a field $F$. If the minimal polynomial $m_T$ is a product of distinct monic irreducible polynomials, show that every $T$-invariant subspace of $V$ has a $T$-invariant complement.
Math 554 Qualifying Exam

August, 2018  Ron Ji

You may use any theorems from the textbook. Any other claims must be proved in details.

1. Let $V$ be the $\mathbb{R}$-vector space consisting of real polynomials in $x$ of degree not exceeding $n$. Let $t_0, t_1, \ldots, t_n$ be distinct real numbers. Define $L_i(f) = f(t_i)$ for $f \in V$ and $i = 0, 1, 2, \ldots, n$. Suppose $I \in V^*$ is defined by $I(f) = \int_0^1 f(x) dx$.
   (a) Show that $\{L_0, L_1, \ldots, L_n\}$ is a basis for the dual space $V^*$.
   (b) Find $P_i$ for $i = 0, 1, \ldots, n$, such that $I = \sum_{i=0}^n P_i L_i$.
   (c) Is the set $\{I, L_1, L_2, \ldots, L_n\}$ a basis for $V^*$?

2. Let $A =\begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$.
   (a) Find the invariant factors, minimal and characteristic polynomials of $A$.
   (b) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$.

3. Let $V$ be a vector space over a field $F$ and $T$ a linear transformation on $V$. Let $\text{Null}(T)$ be the null space of the linear operator $T$ on $V$. Suppose that $p_1, p_2, \ldots, p_n$ are mutually relatively prime polynomials in $F[x]$ and $p = p_1p_2 \cdots p_n$. Show that $\text{Null}(p(T))$, is the direct sum of $\text{Null}(p_i(T))$, for $i = 1, 2, \ldots, n$.

4. Show that for any matrix $A$ in $M_3(\mathbb{C})$,
   \[ A^3 - \text{Tr}(A)A^2 + \text{Tr(adj}A)A - \det(A)I = 0, \]
   where $\text{Tr}(A)$ is the trace of the matrix $A$ and $\text{adj}(A)$ is the classical adjoint matrix of $A$.

5. Show that if both $A$ and $B$ are positive definite $n \times n$ matrices over the real numbers, then $\det(A + B) > \det A + \det B$. 
1. Let \( D \neq I \) be an \( n \times n \) complex matrix such that \( D^3 = I \). Compute the inverse of the matrix \( 2I - D \), or explain why \( 2I - D \) is not invertible.

2. Let \( W \) be the subspace of \( \mathbb{R}^4 \) spanned by the vectors \((1, -1, 1, -1)\) and \((1, 2, 1, 2)\).

   (Use the standard (Euclidean) inner product on \( \mathbb{R}^4 \).)

   (a) Find the matrix for the orthogonal projection of \( \mathbb{R}^4 \) onto \( W \).

   (b) Find the point of \( W \) that is closest to the point \((3, -2, 1, -1)\).

3. Let \( P \) be the vector space of polynomials of degree 3 or less with real coefficients and let \( B = \{1, x, x^2, x^3\} \) be the standard basis for \( P \).

   Define linear functionals, for \( p \) in \( P \), by \( f_0(p) = p(0) \), \( f_1(p) = p(1) \), \( f_2(p) = p(2) \), and \( f_3(p) = p(-1) \). For example, \( f_0(x^3) = 0 \), \( f_1(x^3) = 1 \), \( f_2(x^3) = 8 \), and \( f_3(x^3) = -1 \).

   Find a basis \( C = \{p_0, p_1, p_2, p_3\} \) for \( P \) such that \( \{f_0, f_1, f_2, f_3\} \) is the dual basis for the basis \( C \), where you write \( p_0, p_1, p_2, \) and \( p_3 \) as linear combinations of the standard basis \( B \).

4. Let \( A \) be an \( m \times n \) matrix and \( B \) be an \( n \times m \) matrix, each with entries from the field \( F \).

   (a) Find conditions on \( m, n, A, \) and \( B \) so that \( AB \) is invertible if and only if your conditions are met. (Note: the conditions should be on \( A \) and \( B \) and their interactions, NOT on \( AB \), so \( \det(AB) \neq 0 \) is not an appropriate answer.)

   (b) Prove the theorem you asserted in part (a), that is, prove that “For \( A \) an \( m \times n \) matrix and \( B \) an \( n \times m \) matrix, \( AB \) is invertible if and only if ‘conditions in (a)’ hold.”

5. The eigenvalues of the matrix \( C = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 3 & 1 & 0 \\
1 & -1 & 2 & 0 \\
-2 & 3 & 1 & -1
\end{pmatrix} \) are 2 and \(-1\).

   Find the Jordan Canonical Form, \( J \), for \( C \) and find a matrix \( U \) so that \( J = U^{-1}CU \).

6. Let \( S \) be an \( n \times n \) complex matrix. It is easily proved that for every polynomial \( p \), the matrix \( p(S) \) commutes with \( S \); if you need to use that fact, you do NOT need to prove it, just use it.

   (a) Prove: If \( S \) is an \( n \times n \) matrix with \( n \) distinct eigenvalues and \( T \) is an \( n \times n \) matrix that commutes with \( S \), then there is a polynomial \( p \) of degree \( n \) or less such that \( T=p(S) \).

   (b) The converse of the general statement above is false: Let \( N = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \)

   Find a matrix \( T \) that commutes with \( N \) and \( T \neq p(N) \) for any polynomial \( p \).

   (You need to prove both statements about the matrices \( T \) you create!)
• Explain/prove your answers for each question in such a way that your reasoning can be followed!!

1. Let $A$ be an $m \times n$ matrix and $B$ be an $n \times m$ matrix, each with entries in the field $\mathbb{F}$.
   Prove: If $AB$ is an invertible matrix, then the columns of $B$ are linearly independent.

2. The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, and $v_4 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ are in $\mathbb{R}^3$.
   Find conditions on vectors $w_1$, $w_2$, $w_3$, and $w_4$ in $\mathbb{R}^4$ so that the statement “There is a linear transformation $T$ mapping $\mathbb{R}^3$ into $\mathbb{R}^4$ such that $Tv_j = w_j$ for $j = 1, 2, 3, 4$ if and only if ···” is true. Prove that your conditions are correct.

3. Let $C$ be an $n \times n$ real matrix with $n \geq 3$.
   (a) For which real polynomials $q$ of degree 2 is the null space $q(C)$ not the zero subspace?
   (b) For which real polynomials $f$ of degree $k \geq 3$ is $f(C)$ invertible?

4. Let $E = \begin{pmatrix} -1 & -1 & -3 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$
   Find the Jordan Canonical Form, $J$, for $E$ and find a matrix $S$ so that $J = S^{-1}ES$.

5. On any vector space $V$, the projections 0 and $I$ are considered to be trivial projections.
   Let $L$ be a linear transformation on $\mathbb{C}^n$ for $n \geq 2$ and let $P$ be a non-trivial projection on $\mathbb{C}^n$.
   (a) Show that the range of $P$ is an invariant subspace for $L$ if and only if $PLP = LP$.
   (b) Show that the range and nullspace of $P$ are both invariant for $L$ if and only if $LP = PL$.
   (c) Find an operator $L$ on $\mathbb{C}^n$ for $n \geq 2$ that does not commute with any non-trivial projection OR prove that for $n \geq 2$ and any $L$ on $\mathbb{C}^n$, there is a non-trivial projection $P$ so that $LP = PL$.

6. (a) Let $H$ be a Hermitian (i.e. self-adjoint) matrix and let $G = H^2$.
    Prove that if $\lambda$ is an eigenvalue of $G$, then $\lambda$ is real and $\lambda \geq 0$.
   (b) Let $K$ be a Hermitian matrix all of whose eigenvalues are non-negative real numbers.
    Prove that there is a Hermitian matrix $H$, all of whose eigenvalues are non-negative real numbers, such that $H^2 = K$.
   (c) The complex matrix $K = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ has non-negative eigenvalues.
      Find all Hermitian matrices $H$, with each eigenvalue non-negative, such that $H^2 = K$. 
Math 554 Qualifying Exam
January, 2017
ID #: _____________
Ron Ji

You may use any theorems from the book. Other results you use must be proved. Make sure to double check your calculations and support your arguments.

1. Let
\[
A = \begin{pmatrix}
3 & 1 & -1 \\
2 & 2 & -1 \\
2 & 2 & 0
\end{pmatrix}.
\]
   (a) (10) Find the invariant factors and Jordan canonical form of \( A \).
   (b) (10) Find a diagonalizable matrix \( D \) and a nilpotent matrix \( N \) such that \( A = D + N \) and \( DN = ND \).

2. (15) Let \( T \) be a linear operator on the finite-dimensional space \( V \) over a field \( F \), let \( R \) be the range and \( N \) be the null space of \( T \). Prove that \( R \) has a \( T \)-invariant complement if and only if \( R \) and \( N \) are independent. (Note: Two subspaces \( U \) and \( W \) are independent if whenever \( u + w = 0 \) with \( u \in U \) and \( w \in W \), then \( u = w = 0 \).)

3. (20) Let \( T \) be a linear transformation on a finite dimensional vector space \( V \) over the field \( F \). Let \( W \) be a proper nontrivial subspace of \( V \). Show that \( \dim(TW) + \dim(N(T) \cap W) = \dim W \) where \( TW \) is the image of \( T \) on \( W \) and \( N(T) \) is the null space of \( T \).

4. Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Let \( W \) be a \( T \)-invariant subspace of \( V \). Let \( W^\perp \) be the orthogonal complement of \( W \).
   (a) (5) Show that \( W^\perp \) is \( T^* \)-invariant.
   (b) (5) If \( T \) is normal and \( W \) is a span of some eigenvectors of \( T \), then \( W^\perp \) is both \( T \) and \( T^* \) invariant. (Note: \( T \) is normal if \( T^*T = TT^* \).)
   (c) (10) If \( T \) is normal and \( W \) is \( T \)-invariant, show that \( W \) is also \( T^* \)-invariant.
   (d) (5) Show that if \( T \) is normal and \( W \) is both \( T \) and \( T^* \) invariant, then \( T|_W \) is normal on \( W \).

5. (20) Let \( T \) be a linear operator on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Prove that \( T \) is self-adjoint if and only if \( \langle To|\alpha \rangle \) is real for every \( \alpha \) in \( V \). (Note: \( T \) is self-adjoint if \( T^* = T \).)
Problem 1. Let $A$ be an $n \times n$ matrix such that $\text{rk}(A) \leq 1$. Show that $\text{tr}(A) + 1 = \text{det}(A+1)$.

Problem 2. Count the largest number of pairwise nonsimilar $7 \times 7$ complex matrices $A$ such that $\text{rk}(A - 2) = 5$ and $f(x) = (x - 2)^3(x - 3)^2$ is
a) the minimal annihilating polynomial of $A$;
b) an annihilating polynomial of $A$.

Problem 3. Let $A, B, C$ be $n \times n$ matrices and let $A$ be invertible. True or false? Justify your answer!
a) $\text{rk}(AB) = \text{rk}(BA)$;
b) $\text{rk}(BC) = \text{rk}(CB)$;
c) $\text{rk}(ABC) = \text{rk}(BAC)$.

Problem 4. Let $A$ be a $3 \times 3$ matrix such that $\text{tr}(A) = \text{tr}(A^2) = \text{tr}(A^3) = 2$. Show that $A$ is not invertible.

Problem 5. What is the largest possible dimension of a subspace $W \subset \mathbb{R}^4$ such that the restriction of the form $Q(x_1, x_2, x_3, x_4) = x_1x_2 + x_2x_3 + x_3x_4$ to $W$ is positive definite?

Problem 6. Let $W_n$ be a subspace of the space $M(n, \mathbb{C})$ of complex $n \times n$ matrices spanned by matrices of the form $[A, B]$, i.e. $W_n = \{AB - BA \mid A, B \in M(n, \mathbb{C})\}$. Show $\dim W_n = n^2 - 1$. 

Throughout this examination, $m$ denotes the Lebesgue measure on $\mathbb{R}$.

Problem 1: Show that there is a compact set $K \subset [0, 1]$ with $m(K) \geq 11/12$, but such that $K$ contains no non-empty open intervals.

Problem 2: Suppose $(f_n)$ is a sequence in $L_1[0, 1]$ such that $\lim_{n \to \infty} \int_0^1 |f_n(x)| \, dm(x) = 0$. Show that for all $\delta > 0$, we have $m\{x \in [0, 1] : |f_n(x)| \geq \delta \} \to 0$ as $n \to \infty$.

Problem 3: Let $f \geq 0$ be a measurable function on $[0, 1]$. Assume $m\{f \geq x\} \leq 1/x^2$ for all $x \geq 1$. Show that $f \in L_r[0, 1]$ for all $1 \leq r < 2$.

Problem 4: Show that for $p, 1 < p \leq \infty$, $L_p[0, 1] \subset L_1[0, 1]$ but $L_p(\mathbb{R})$ is not a subset of $L_1(\mathbb{R})$.

Problem 5: Let $(r_n)$ be the sequence of functions such that $r_n : [0, 1) \to \{-1, 1\}$ given by the rule that $r_n(x) = (-1)^k$ for $x \in [\frac{k}{2^n}, \frac{k+1}{2^n})$ with $k = 0, \ldots, 2^n - 1$. Let $f \in L_1[0, 1]$. Show that $\lim_{n \to \infty} \int_0^1 f(x)r_n(x) \, dm(x) = 0$.

Problem 6: Show that for $f \in L_1(\mathbb{R})$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f(x + 1/n) - f(x)| \, dm(x) = 0.$$
1. (20) Let $A \in M_{m \times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Prove the following.
   
   (a) The equation $AX = b$ has a solution if and only if $b \in \ker(A^T)^\perp$.
   
   (b) $A$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if $A^T X = 0$ has only trivial solution.
2. (20) Let \( A = (a_{ij}) \) be an \( n \times n \) matrix with complex entries. Show that if

\[
|a_{ii}| > \Sigma_{j=1, (j \neq i)}^{n} |a_{ij}|
\]

for \( i = 1, 2, ..., n \), then \( A \) is invertible.
3. (20) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Show that the following are equivalent.

(a) $T$ has a cyclic vector;

(b) If $TS = ST$ for some operator $S$ on $V$, then $S = g(T)$ for some polynomial $g$ over $F$. 
4. (20) Let $T : V \rightarrow V$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Show that the following are equivalent.

(a) Every $T$-invariant subspace of $V$ has a $T$-invariant complement.

(b) The minimal polynomial $p_T$ of $T$ is a product of distinct monic and irreducible polynomials in $\mathbb{F}[x]$:
5. (20) Let $A = (a_{ij})$ be an $n \times n$ matrix with complex entries. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$ counting multiplicity. Show that

$$\sum_{i,j} |a_{ij}|^2 \geq \sum_{i=1}^n |\lambda_i|^2.$$ 

Moreover, equality holds if and only if $A$ is normal. (Hint: $\sum_{i,j} |a_{ij}|^2 = \text{Tr}(A^*A)$.)
Problem 1. Let $V$ be a vector space. Let $\{v_1, v_2, \ldots, v_k\}$ and $\{w_1, w_2, \ldots, w_l\}$ be two linearly independent sets of vectors in $V$. Assume $l > k$. Show that there exists $i \in \{1, \ldots, l\}$ such that the set $\{v_1, v_2, \ldots, v_k, w_i\}$ is linearly independent.

Problem 2. Let $V_n = \text{Mat}_R(n, n)$ be the space of real matrices of size $n \times n$. Let $A_n, U_n, D_n \subset V$ be the subspaces of skew-symmetric, strictly upper triangular matrices, and diagonal matrices respectively.

a) Show that $V_n = U_n \oplus A_n \oplus D_n$ is a direct sum decomposition.

b) Find the projection of the matrix $M$ to $U_3$ along $A_3 \oplus D_3$.

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} \in V_3.$$ 

Problem 3. We say that $n \times n$ complex matrices $A$ and $B$ are equivalent if there exist invertible $n \times n$ complex matrices $X, Y$ such that $XAY = B$. In such a case we write $A \sim B$.

a) Show that $\sim$ is an equivalence relation.

b) True or false? If $A \sim B$ then $A^{-1} \sim B^{-1}$. Explain!

c) True or false? If $A \sim B$ then $A^2 \sim B^2$. Explain!

Problem 4. Consider the $n \times n$ matrix $A = (\delta_{i,n+1-i}a_i)$, where $a_i \in \mathbb{C}$:

$$A = \begin{pmatrix} 0 & 0 & \ldots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1} & \ldots & 0 \\ a_n & 0 & \ldots & 0 \end{pmatrix}.$$ 

Give the minimal and characteristic polynomials of $A$. Give a sufficient and necessary condition for $A$ to be diagonalizable. Describe the Jordan canonical form of $A$.

Problem 5. Let $C : \mathbb{C}^4 \to \mathbb{C}^4$ be a linear operator given in basis $\{e_1, e_2, e_3, e_4\}$ by the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$ 

Describe all possible bases (in terms of $e_1, e_2, e_3, e_4$) in which $C$ is given by a matrix in a Jordan canonical form.

Problem 6. Let $B(x, y)$ be a bilinear form on $\mathbb{R}^{n+k}$ given by the formula

$$B(x, y) = \sum_{i=1}^{n} x_i y_i - \sum_{j=1}^{k} x_{n+j} y_{n+j}, \quad x, y \in \mathbb{R}^{n+k}.$$ 

What is the maximal dimension of the subspace $W \subset \mathbb{R}^{n+k}$ such that $B(x, y) = 0$ for all $x, y \in W$? Give an example of such a subspace.
1. (20) Let

\[ A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}. \]

(a) Find the invariant factors and minimal polynomials of \( A \);
(b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).

(Note: \( A \) has only one eigenvalue of multiplicity 4.)
2. (20) Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{Q}$ with the same minimal polynomial and the same characteristic polynomial.

(a) Prove that if $n = 3$, then $A$ and $B$ are similar.

(b) Prove or disprove that if $n = 4$, then $A$ and $B$ are similar.
3. (20) It is known that for an $n \times n$ matrix $A$ over $\mathbb{R}$ the classical adjoint $\text{adj} A$ of $A$ satisfying:

$$\text{rank}(\text{adj} A) = \begin{cases} n & \text{if } \text{rank } A = n \\ 1 & \text{if } \text{rank } A = n - 1 \\ 0 & \text{if } \text{rank } A < n - 1. \end{cases}$$

For $n \geq 2$, show that

(a) $\det(\text{adj} A) = (\det A)^{n-1}$;

(b) $\text{adj}(\text{adj} A) = (\det A)^{n-2} A$. 
4. (20) Let $W$ be an invariant subspace of a matrix $A \in M_{n \times n}(\mathbb{R})$. Let $f_A$ be the characteristic polynomial of $A$. Prove the following.

(a) $W^\perp$ is invariant under $A^T$, where $A^T$ is the transpose matrix of $A$.

(b) $f_A(x) = f_{A|W}(x)f_{A^T|W^\perp}(x)$, where $A|W$ is the restriction of the linear transformation $A$ to $W$. 
5. (20) Recall that an \( n \times n \) matrix \( A \) over \( \mathbb{C} \) is normal if \( A^* A = AA^* \), where \( A^* = \bar{A}^T \). Show that if \( A \in M_{3 \times 3}(\mathbb{R}) \) is normal, then there is an orthogonal matrix \( O \) such that

\[
O^T A O
\]

is either diagonal or is in the form

\[
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & a & b \\
0 & -b & a
\end{pmatrix},
\]

where \( b \neq 0 \).
1. (20) Let
\[
A = \begin{pmatrix}
-7 & -1 & -1 \\
-21 & -3 & -3 \\
70 & 10 & 10
\end{pmatrix}.
\]

Find the Jordan canonical form \(J\) of \(A\) and an invertible matrix \(P\) such that \(P^{-1}AP = J\).
2. Let $T$ be a linear operator on an $n$-dimensional vector space $V$ ($n > 1$) and $W$ is a $k$-dimensional ($0 < k < n$) $T$-invariant subspace. Show that if $T$ has $n$ distinct eigenvalues, then for any $T$-invariant direct sum decomposition of $V = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, $W = (W_1 \cap W) \oplus (W_2 \cap W) \oplus \cdots \oplus (W_s \cap W)$. 
3. (20) Let $T$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. Let $p_T$ and $f_T$ be the minimal and respectively characteristic polynomial of $T$. If $p_T = f_T = q^k$ for some irreducible polynomial $q \in \mathbb{F}[x]$ and $k > 1$, show that no nonzero proper $T$-invariant subspace can have a $T$-invariant complement.
4. (20) Let $A$ be the $2n \times 2n$ complex matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_{2n} \\
0 & 0 & \cdots & a_{2n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_2 & \cdots & 0 & \cdots \\
a_1 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
$$

Find a necessary and sufficient condition that the matrix $A$ is diagonalizable. Justify your answer.
5. (20) Recall that an operator $N$ on a finite dimensional inner product space is normal if $N^*N = NN^*$. Show that the product $ST$ of two normal operators $S$ and $T$ on a finite dimensional inner product space $V$ is normal if $ST = TS$. 
Math 554 Qualifying Exam

January, 2013

Make sure to provide detailed arguments to support your claims! For each problem you may prove one part by using the claims of the other parts.

1. Let \( A = \begin{pmatrix} -5 & 1 & 1 \\ -3 & -1 & 1 \\ -6 & 2 & 0 \end{pmatrix} \)
   a) (10) Find the characteristic polynomial and minimal polynomial of \( A \).
   b) (5) Determine if \( A \) is diagonalizable or not.
   c) (5) Find the rational form of \( A \).
   d) (5) Find the Jordan canonical form \( J \) of \( A \).
   e) (10) Find an invertible matrix \( P \) such that \( P^{-1}AP = J \).

(Make sure to double check part a)!

2. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the linear transformation defined by reflecting a vector in \( \mathbb{R}^2 \) with respect to the line \( y - \sqrt{3}x = 0 \) followed by reflecting the resulting vector with respect to the line \( y + x = 0 \).
   a) (10) Find a matrix representation of \( T \) with respect to the standard basis \( E = \{ (1, 0), (0, 1) \} \).
   b) (5) Show that \( T \) is a (counter-clockwise) rotation and find the angle of the rotation.

3. Let \( T \) be a linear operator on an \( n \)-dimensional vector space \( V \). Let \( R(T) \) be the range of the operator \( T \) on \( V \) and \( N(T) \) be the null space of \( T \).
   a) (7) Let \( R_\infty(T) = \cap_{k=1}^\infty R(T^k) \). Show that \( R_\infty(T) = R(T^n) \).
   b) (8) Let \( N_\infty(T) = \cup_{k=1}^\infty N(T^k) \). Show that \( N_\infty(T) = N(T^n) \).

4. (15) Let \( S \) and \( T \) be two commuting linear operators on a finite dimensional vector space \( V \). If the minimal and characteristic polynomials of \( T \) are equal, show that \( S \) is a polynomial in \( T \).

5. Let \( A \) be an \( n \times n \) antisymmetric matrix: \( A^T = -A \) over \( \mathbb{R} \).
   a) (10) Show that there is an invertible \( n \times n \) matrix \( P \) such that \( PAP^T \) is equal to the block diagonal matrix \( J = \text{diag}\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \} \), where the last 0 in the matrix is a square 0 matrix of certain dimension. (Hint: use both elementary row and column operations.)
   b) (10) Show that there are symmetric matrices \( C \) and \( D \) such that \( A = CD - DC \).
Math 554 Qualifying Exam

August, 2013

Name: __________________

Make sure to double check your calculations!

1. (10) Show that if $f$ is a polynomial in $\mathbb{C}[x]$ of degree $n \geq 2$, then $f'|f$ if and only if $f(x) = a(x - c)^n$ for some $c \in \mathbb{C}$. 


2. (15) Let $adjA$ be the classical adjoint of an $n \times n$ matrix $A$ over $\mathbb{R}$. Show that

$$\text{rank}(adjA) = \begin{cases} 
  n & \text{if } \text{rank}A = n \\
  1 & \text{if } \text{rank}A = n - 1 \\
  0 & \text{if } \text{rank}A < n - 1.
\end{cases}$$
3. (15) Let $A$ be the $n \times n$ matrix
\[
\begin{pmatrix}
  x & a & a & a & \cdots & a \\
  -a & x & a & a & \cdots & a \\
  -a & -a & x & a & \cdots & a \\
  -a & -a & -a & x & \cdots & a \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  -a & -a & -a & -a & \cdots & x \\
\end{pmatrix}
\]. Find the determinant of $A$. 
4. (10) If $A$ is an invertible matrix over $\mathbb{R}$, show that there are positive constants $c_1 < c_2$ such that

$$c_1 X^T X \leq X^T A^T A X \leq c_2 X^T X,$$

for all $X \in \mathbb{R}^{n \times 1}$. 
5. (10) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be linearly independent in a vector space $V$ over the field $\mathbb{F}$. Assume that $A$ is an $n \times k$ matrix over $\mathbb{F}$ and

$$(\beta_1, \beta_2, \ldots, \beta_k) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \cdot A.$$ 

Show that the dimension of the subspace $W =: \text{span}\{\beta_1, \beta_2, \ldots, \beta_k\}$ is the rank of $A$. 
6. (10) Find the minimal polynomial of the matrix

\[ A = \begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}. \]
7. (15) Find the Jordan form $J$ of the matrix $A = \begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}$, and find an invertible matrix $P$ such that $P^{-1}AP = J$. 
Math 554 Qualifying Exam

January, 2012

Ron Ji

Make sure to provide necessary reasonings for all your claims!

1. (15) Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{R}$ which are similar over $\mathbb{C}$. Prove or disprove that $A$ is similar to $B$ over $\mathbb{R}$. 
2. (15) Let $A$ be an $n \times n$ matrix with entries in $\mathbb{C}$ such that for each eigenvalue $c$ of $A$, the eigensubspace of $A$ associated with $c$ is one dimensional. Let $B$ be any $n \times n$ matrix satisfying $AB = BA$. Show that there is a polynomial $f \in \mathbb{C}[x]$ such that $B = f(A)$. 
3. (15) Let $V$ be a finite dimensional vector space over the field $F$ and $T$ be a linear transformation on $V$. For any vector $\alpha \in V$, let $Z(\alpha; T) = \{f(T)\alpha | f \in F[x]\}$. Suppose that $V$ has two cyclic decompositions:

$V = Z(\beta_1; T) \oplus Z(\beta_2; T)$, with the $T$-annihilator of $\beta_i$ being $q_i$ and satisfying $q_1$ divides $q_2$;

$V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_r; T)$, with the $T$-annihilator of $\alpha_i$ being $p_i$ satisfying $p_i$ divides $p_{i+1}$ for $i = 1, 2, ..., r - 1$.

Show that $r = 2$, $p_1 = q_1$ and $p_2 = q_2$. 
4. (30) Let \( A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 35 & 21 & 1 & 0 \\ -15 & -9 & 0 & 1 \end{pmatrix} \).

a) Find the invariant factors, minimal polynomial and the characteristic polynomial of \( A \).

b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
5. (15) Let $T$ be a linear operator on the $n$-dimensional inner product space $V$ over $\mathbb{C}$ satisfying the property that $TT^* = f(T)$, where $f$ is any polynomial in $\mathbb{C}[x]$ such that $f(0) = 0$. Show that $T$ is normal. (Recall that the adjoint operator $T^*$ is defined by the equality $(T\alpha|\beta) = (\alpha|T^*\beta)$ for all vectors $\alpha$ and $\beta$ in $V$ and $T$ is normal if $TT^* = T^*T$.)
1. (20) Prove that the determinant of the $n \times n$ matrix

$$D_n = \begin{pmatrix}
    a & b & b & \cdots & b \\
    c & a & b & \cdots & b \\
    c & c & a & \cdots & b \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c & c & c & \cdots & a
\end{pmatrix}$$

is $\frac{c(a-b)^n - b(a-c)^n}{c-b}$ for $b \neq c$.

2. (20) Let $E_1, E_2, \ldots, E_n$ be projections on a vector space $V$ over a field $F$ of characteristic 0, such that, for each $k$ with $1 \leq k \leq n$, $E_1 + E_2 + \cdots + E_k$ is also a projection. Prove that $E_iE_j = E_jE_i = 0$ for all $1 \leq i < j \leq n$.

3. (20) Let $A$ be an invertible $n \times n$ matrix over the field of complex numbers. Show that

   a) $A$ and $A^{-1}$ have the same eigensubspaces.

   b) If $J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$, where $J_i = \begin{pmatrix} c_i & 1 & & \\ & c_i & & \\ & & \ddots & \\ & & & c_i \end{pmatrix}_{k_i \times k_i}$, then the Jordan form for $A$ is $J'$, where

$$J' = \begin{pmatrix} J'_1 & & \\ & \ddots & \\ & & J'_k \end{pmatrix},$$

with $J'_i = \begin{pmatrix} c_i^{-1} & 1 & & \\ & c_i^{-1} & & \\ & & \ddots & \\ & & & c_i^{-1} \end{pmatrix}_{k_i \times k_i}$ for each $i$. 

Make sure to provide detailed arguments to support your claims!
4. (20) Recall that a linear transformation $T$ on a finite dimensional inner product space $V$ over $\mathbb{C}$ is normal if and only if $T^*T = TT^*$. Show that the following statements are equivalent for a linear transformation $T$ on $V$.
   (a) $T$ is normal on $V$;
   (b) $\|T\alpha\| = \|T^*\alpha\|$ for every vector $\alpha \in V$;
   (c) $T^*$ is a polynomial (with complex coefficients) in $T$.

5. (20) Let $T$ be a linear transformation on the finite-dimensional vector space $V$, let $p = p_1^{r_1} \cdots p_k^{r_k}$ be the minimal polynomial for $T$ (where $r_i > 0$), and let $V = W_1 \oplus \cdots \oplus W_k$ be the primary decomposition for $T$, i.e. $W_j$ is the null space of $p_j(T)^{r_j}$. Let $W$ be any $T$-invariant subspace of $V$. Show that $W = (W \cap W_1) \oplus (W \cap W_2) \oplus \cdots \oplus (W \cap W_k)$. 

1. (20) Let $W_1$ and $W_2$ be subspaces of a finite dimensional inner product space $V$. Show that

(a) $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$;

(b) $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$, where $W^\perp$ denotes the orthogonal complement of the subspace $W$ in $V$. 

Make sure to double check your solutions!
2. (20) Let $A = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
4 & 3 & 0 & 0 \\
35 & 21 & 1 & 0 \\
-15 & -9 & 0 & 1
\end{pmatrix}$.

a) Find the invariant factors, minimal polynomial and the characteristic polynomial of $A$.

b) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$. 

3. (20) Let $T : V \rightarrow V$ be a linear transformation on the finite dimensional vector space $V$ over a field $F$. If $T^m = T$ for some positive integer $m > 1$, show that

(a) $\text{Null}(T) \cap \text{Im}(T) = \{0\}$, where $\text{Null}(T)$ and $\text{Im}(T)$ are respectively the null space of $T$ and the image of $T$;

(b) $\text{Null}(T) = \text{Null}(T^k)$ for any positive integer $k$. 
4. (20) Let \( T : V \rightarrow V \) be a linear transformation on the finite dimensional vector space \( V \) over \( \mathbb{C} \). Suppose that \( T^3 + 3T = I \), show that \( T \) is diagonalizable.
5. (20) Let $T : V \to V$ be a linear transformation on the finite dimensional vector space $V$ over a field $F$. Let $W$ be a nonzero proper $T$-invariant subspace of $V$. Suppose that the characteristic polynomial $f_T$ of $T$ satisfies that $f_T(0) \neq 0$. Show that if $T$ has a cyclic vector, then $T|_W : W \to W$ has a cyclic vector.
Math 554 Qualifying Exam

August, 2010  Ron Ji

You may use any theorems from the textbook. Any other claims must be proved in
details.

1. (20) Let $V$ be the $\mathbb{R}$-vector space consisting of real polynomials of degree not exceeding $n$. Let $(t_0, t_1, ..., t_n)$ and $(t'_0, t'_1, t'_2, ..., t'_n)$ be $(n + 1)$-tuples of distinct real numbers. Define $L_i(f) = f(t_i)$ and $L'_i(f) = f(t'_i)$ for $f \in V$ and $i = 0, 1, 2, ..., n$.

(a) Show that both sets $\{L_0, L_1, ..., L_n\}$ and $\{L'_0, L'_1, ..., L'_n\}$ are bases for the dual space $V^*$.

(b) Find an $n \times n$ invertible real matrix $P = (P_{ij})$ (in terms of $t_i$'s and $t'_j$'s) such that $L'_j = \sum_{i=0}^{n} P_{ij} L_i$. 
2. (30) Let \( A = \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 3 \end{pmatrix} \).

(a) Find the invariant factors, minimal and characteristic polynomials of \( A \).

(b) Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
3. (15) Let $T$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. If the minimal polynomial of $T$ is irreducible, prove the following.

(a) Every $T$-invariant subspace $W$ of $V$ is $T$-admissible, that is, for any polynomial $f \in \mathbb{F}[x]$ if $f(T)\alpha \in W$, then there is $\beta \in W$ such that $f(T)\alpha = f(T)\beta$.

(b) Every $T$-invariant subspace $W$ has a $T$-invariant complement $W'$ satisfying: (i) $W + W' = V$, and (ii) $W \cap W' = \{0\}$.
4. (15) (a) Suppose that $A$ and $B$ are $3 \times 3$ matrices over the field $F$. Show that $A$ is similar to $B$ if and only if $A$ and $B$ have the same minimal polynomial and the same characteristic polynomial. (b) Is the statement in (a) true or not in general?
5. (10) Let $\mathbb{F}$ be a field. Suppose that $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{n \times m}(\mathbb{F})$. Show that

(a) $\det(I_m - AB) = \det(I_n - BA)$.

(b) if $n \leq m$, then $\det(xI_m - AB) = x^{m-n}\det(xI_n - BA)$. 
6. (20) If each $A_i$ is symmetric $n \times n$ real matrix and $\sum_{i=1}^{m} A_i = I_n$, where $I_n$ is the $n \times n$ identity matrix, then the following conditions are equivalent:

(1) $A_i^2 = A_i$, for $i = 1, 2, 3, \ldots, m$;
(2) $A_i A_j = 0$, whenever $1 \leq i, j \leq m$ and $i \neq j$;
(3) $\sum_{i=1}^{m} \text{rank} A_i = n$.

(Hint: For certain directions the trace function or induction might help.)
Math 554 Qualifying Exam

Fall, 2009

Ron Ji

1. (20) Let $V$ be the vector space of $n \times n$ matrices over the field $\mathbb{F}$ and $V_0$ be the subspace consisting of matrices of the form $C = AB - BA$ for some $A, B$ in $V$. Prove that $V_0 = \{ A \in V | \text{Trace}(A) = 0 \}$.

2. (15) If $W$ is a subspace of a finite dimensional vector space $V$ and if $\{ g_1, g_2, \ldots, g_r \}$ is a basis of $W^0 = \{ f \in V^* | f|_W = 0 \}$, then $W = \bigcap_{i=1}^r N_{g_i}$, where for $f \in V^*$, $N_f = \{ \alpha \in V | f(\alpha) = 0 \}$.

3. (25) Let $\langle A|B \rangle = \text{Trace}(AB^*)$ be the inner product on $M_n(\mathbb{C})$, the $n \times n$ matrices over the field $\mathbb{C}$. Let $A$ be in $M_n(\mathbb{C})$ and $T_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the linear transformation defined by $T_A(B) = ABA^*$, where $A^* = \bar{A}^t$. Show that
   (1) $T_A$ is invertible if and only if $A$ is invertible.
   (2) $T_A$ is unitary if and only if $A$ is unitary.
   (3) $T_A$ is self-adjoint if and only if $A$ is normal and $\bar{\lambda} \bar{\mu}$ is real for any two eigenvalues $\lambda$ and $\mu$ of $A$.
   (Hint: If $T_A$ is self-adjoint, show that $A$ is normal first; and if $A$ is normal, there is a unitary matrix $U$ and a diagonal matrix $D$ such that $UDU^* = A$.)

4. (20) Let $T$ be a linear transformation on the finite dimensional vector space $V$ over the field $\mathbb{F}$. Let $p$ be the minimal polynomial of $T$. If $p = g_1g_2$, where $g_1$ and $g_2$ are relatively prime factors of $p$, show that
   (a) $V = W_1 \oplus W_2$, where $W_i = \{ \alpha \in V | g_i(T)\alpha = 0 \}$, for $i = 1, 2$.
   (b) $W_1$ and $W_2$ are $T$-invariant.
   (c) If $T_i$ is the operator on $W_i$ induced by $T$, then the minimal polynomial for $T_i$ is $g_i$ for $i = 1, 2$.

5. (20) Let $A = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.
   a) Find the minimal polynomial and the characteristic polynomial of $A$.
   b) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$. 
Any points over 100 will be bonus. You may use any theorems stated in the textbook or proved in the lectures.

1. (10) Let $V$ be an $n$-dimensional vector space over a field $F$. Show that the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a basis for $V$ if and only if for any nonzero linear functional $f$ on $V$, there is a nonzero vector $\alpha$ in the span of $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $f(\alpha) \neq 0$.

2. (15) Let $W_1$ and $W_2$ be subspaces of a finite dimensional inner product space $V$. Show that

$$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp,$$

where $W^\perp$ denotes the orthogonal complement of the subspace $W$ in $V$.

3. Let $A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$.

   a) (10) Find the minimal polynomial and the characteristic polynomial of $A$.
   b) (15) Find the Jordan canonical form $J$ of $A$ and an invertible matrix $P$ such that $P^{-1}AP = J$.

4. (20) Suppose that $A$ and $B$ are $n \times n$ matrices such that $AB = 0$. Show that $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \leq n$.

5. (20) Let $T$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. Let $c$ be a characteristic value of $T$ and $W_c$ be the characteristic subspace of $T$ associated with $c$. Suppose that a proper $T$-invariant subspace $W$ contains $W_c$, and there is a vector $\alpha$ in $V$ but not in $W$ such that $(T - cI)\alpha$ is in $W$. Show that the minimal polynomial $p_T$ of $T$ is in the form $(x - c)^2q$ for some nonzero polynomial $q$ in $F[x]$.

6. (20) Let $T$ be a linear transformation on a finite dimensional vector space $V$ over a field $F$. If the minimal polynomial $p_T$ of $T$ is irreducible, then every $T$-invariant subspace $W$ has a $T$-invariant complement $W'$. That is, $TW \subset W$, $TW' \subset W'$, $W + W' = V$ and $W \cap W' = \{0\}$. 