Problem 1

Let \( \mathbb{Q} \) denote the set of rational numbers. Find all functions \( f : \mathbb{Q} \to \mathbb{Q} \) such that

A) for every \( x, y \) and \( w \) we have \( f(x + y - w) = f(x) + f(y) - f(w) \).

B) for every \( x, y, z \) and \( w \) we have \( f(x + y + z - 2w) = f(x + y - w) + f(y + z - w) + f(x + z - w) - f(x) - f(y) - f(z) + f(w) \).

A) Notice that if two functions satisfy the desired identity, then so do their sum and difference. Also notice that linear functions \( bx + c \), where \( b, c \in \mathbb{Q} \), do satisfy the identity. Therefore, if \( f(x) \) satisfies the identity, then so does

\[
g(x) = f(x) - (f(1) - f(0))x - f(0), \quad g(1) = g(0) = 0.
\]

Then, setting \( w = 0 \) in the identity gives \( g(x + y) = g(x) + g(y) \), i.e., \( g(x) \) is linear; and setting \( y = -x \) and \( w = 0 \) also gives \( 0 = g(x) + g(-x) \), i.e., \( g(x) \) is odd. Hence,

\[
g(nx) = n \cdot g(x) \text{ for all integers } n.
\]

Now, if we represent \( x \in \mathbb{Q} \) as \( x = n/m \), where \( n, m \) are integers, then

\[
mg(n/m) = g(n) = ng(1) \implies g(x) = xg(1) = 0 \text{ for any } x \in \mathbb{Q}.
\]

Thus, only linear functions \( bx + c \), \( b, c \in \mathbb{Q} \), satisfy the identity.

B) Again, notice that if two functions satisfy the desired identity, then so do their sum and difference. Also observe that quadratic functions \( ax^2 + bx + c \), where \( a, b, c \in \mathbb{Q} \), do satisfy the identity. Therefore, if \( f(x) \) satisfies the identity, then so does

\[
g(x) = f(x) - \left( \frac{f(1) + f(-1)}{2} - f(0) \right) x^2 - \frac{f(1) - f(-1)}{2} x - f(0),
\]

for which it holds that \( g(-1) = g(1) = g(0) = 0 \). Let

\[
h_v(x) = g(x + v) - g(x), \quad v \in \mathbb{Q}.
\]

Using the identity with \( x = y, z = w + v, \) and \( w = w \) shows that \( h_v(x) \) satisfies the identity from part A. Hence, it is a linear function and therefore is given by

\[
h_v(x) = (h_v(1) - h_v(0))x + h_v(0) = (g(1 + v) - g(v))x + g(v).
\]

Analogously, \( g(v + 1) - g(v) = h_1(v) = 0 \) for all \( v \in \mathbb{Q} \) since it is a linear function that vanishes at 0 and \(-1\). Therefore,

\[
h_v(x) = g(v) \implies g(x + v) = g(x) + g(v).
\]

In other words, \( g(x) \) is a linear function that vanishes at three points, namely \(-1, 0, 1\), and therefore \( g(x) = 0 \) for all \( x \in \mathbb{Q} \). Hence, only quadratic functions \( ax^2 + bx + c \), \( a, b, c \in \mathbb{Q} \), satisfy the identity.
Problem 2

A toy racetrack comes with an unlimited number of pieces of road, each of which consists of a perfect 90 degree turn attached to a 1 foot by 1 foot square piece of plastic.

A) Determine the 5 shortest closed racetracks that can be formed using only these pieces of road. The road is not allowed to go over itself or cross itself. Different configurations of the same length should be listed.

B) Determine all possible lengths of closed racetracks.

B) If one precedes clockwise along the track the number of right turns must be equal to the number of left turn plus 4. Hence, the length must be even. Since in each square the track enters vertically and exits horizontally or vice versa, the track can be split into pairs of horizontally adjacent squares with the track going from one to the other. The track leaves any such pair only through the top or bottom edges. These pairs are of two types: a) both turns are in the same direction; b) turns are in opposite directions. Pairs a) move the track horizontally by length 1 and pairs b) move the track in both directions (horizontally and vertically) by length 1. Since the track is closed, the total increment in both directions must be 0. Thus, there must be even number of b)-pairs and respectively even number of a)-pairs. That is, the total length must be divisible by 4.

Moreover, if the track has at least two b)-pairs, it needs more than two more pairs to close, and therefore there are no tracks of length 8. Finally, there is a configuration of length 4 and any configuration of length 4n, \( n \geq 3 \), can be achieved by a move as on the first figure (replace the red part of length 3 by the blue part of length 7).

A) The shortest tracks are the 4-turn track, the 12-turn track (black+red on the figure), 16-turn track as on the figure (black+blue on the figure), and two 20-turn tracks obtained from the 16-turn one by the move described above and one 20-turn track as on the last figure.
Problem 3

A master locksmith is designing a combination lock for the garage at the Indy 500. Since 33 cars compete in the Indy 500, the locksmith decides to use 33 different buttons on the keypad: 0,1,...,9,A,B,...,W. A combination consists of pressing a sequence of buttons such that no button is pressed more than once, but different combinations may be created by pressing some buttons simultaneously. For example, using just the first three buttons (0, 1, 2) there are 25 combinations - 0, 1, 2, 01, 10, 02, 20, 12, 21, 012, 021, 102, 120, 201, 210, (01), (02), (12), (01)2, (02)1, (12)0, 2(01), 1(02), 0(12), and (012) - pressing all three buttons simultaneously. To complete the job, the locksmith need to know the total number of possible combinations, call it $C$. While stuck on this problem, the locksmith decides to attend the race. Inspired by the sizzling action, he quickly computes the number of possible outcomes for the race, including possible ties. (Every car finishes.) Call this number of outcomes $R$. On the way back to work, the locksmith discovers a startling relationship between the numbers $R$ and $C$. What is it?

Label the cars as buttons on the lock’s keypad by 0,1,...,9,A,B,...,W. Then, on the one hand, every race ordering is in one-to-one correspondence with the lock combinations where each button was pressed; and on the other hand, every race ordering except for the one where they all tie for the first place, is in one-to-one correspondence with the lock combinations where at least one button was not pressed with omitted buttons corresponding to the cars finishing last. Hence, $C = 2R − 1$. 
Problem 4

Within a $3 \times 2018$ rectangle, each $1 \times 1$ square is colored either red or blue. There are $2^{3 \cdot 2018}$ total configurations possible; of these, a relatively small subset are colored in such a way that no two blue squares are adjacent either horizontally, vertically, or diagonally. Determine the last 6 digits of this smaller number.

Solution A. There are only five possible color combinations for each column: RRR, RRB, BRR, and BRB. Denote by $w_1(n), w_2(n), w_3(n), w_4(n),$ and $w_5(n)$ the number of ways to color a $3 \times n$ rectangle such that the last column is RRR, RRB, BRR, RBR, and BRB, respectively. Then

$$w_1(n+1) = w_1(n) + w_2(n) + w_3(n) + w_4(n) + w_5(n)$$

since the column RRR can be preceded by any allowed column. The column BRR/RRB can be preceded only by RRB/BRR and RRR. Hence, $w_2(n+1) = w_3(n) + w_1(n)$ and $w_5(n+1) = w_5(n) + w_1(n)$. Since $w_2(1) = w_3(1) = 1, w_2(n) = w_3(n)$ for all $n$. That is, $w_2(n+1) = w_2(n) + w_1(n)$. Finally, the columns RRR and RRB can be preceded only by RRR, we have that $w_4(n) = w_5(n) = w_1(n-1)$. Altogether, $w_1(1) = w_2(1) = 1$ and

$$w_1(n+1) = w_1(n) + 2w_2(n) + 2w_1(n-1)$$
$$w_2(n+1) = w_2(n) + w_1(n)$$

Solving this double recursion on a computer gives an answer 800955.

Solution B. Denote the number of possible configurations on $3 \times n$ board by $c(n)$. To compute $c(n+1)$, notice that if the $n$-th column is RRR, then we have all 5 options for the $(n+1)$-st column. Thus, the contribution will be $5c(n-1)$. If the $n$-th column is BRR / RBR, then $(n+1)$-th column must be RRR and for the $(n+1)$-st column we have only one option RRR. Hence, the corresponding contribution is $2c(n-2)$. If the $n$-th column is BRR / RBR, then $(n+1)$-column has two options: RRR & RRB / RBR & BRR. What precedes such a $n$-th column is a block of alternating BRR and RRB columns, which either takes over the whole rectangle or ends with RRR column. Hence, the corresponding contribution is $4(4c(n-2)+c(n-3)+\cdots+c(1)+1)$. Altogether

$$c(n+1) = 5c(n-1) + 2c(n-2) + 4s(n-2), \quad s(k) = 2 + c(1) + \cdots + c(k).$$

In case you were wondering, the actual number is

62093508363312627481223573471375399344278227748238232134048920945322113038413275049
167525675086699003768458458538914818139702848377141295883221449965828315131577683744
1404119538777251680759095735377804256708946866938181492407983795256264024129414459
2086506788844862363497247802742143535904508626967988572869593757979793985074716150
2080766894201314082190210070144594463127908396124239203131445066826824905378479494
70260536798745898474263343558146950420561161053772820321376297320694289505053647251
9745273031207719803099446050892653920185142608066681280842495077864482724952811471
0873428272121718067390403169063075136475894859918650423744854454330533915444682702
64095490265805075998516952852530055464614030672709077370225178039333745907592540585
22182224435757599468767410654977414246856868711678386433500053244773880491425470975
35792148233607446848141462136653259075603123998636271647217507894395326800955.
**Team Problem**

Let us define a semi-multiperfect number as a number such that the sum of all divisors of the number, including the number itself, equals $n + 1/2$ times the number, where $n$ is an integer that we call the genus of the number. For example, 24 is semi-multiperfect of genus 2 because $1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60 = (2 + 1/2) \cdot 24$.

A) Find all semi-multiperfect numbers of genus 3 that are less than $10^9$.

B) Find all semi-multiperfect numbers of genus 4 that are less than $10^{12}$.

C) Find all semi-multiperfect numbers of genus 5 that are less than $10^{15}$.

A) 4320, 4680, 26208, 20427264, 197064960

B) 8910720, 17428320, 8583644160, 57629644800, 206166804480

C) 17116004505600

Generally speaking, this problem is an application of the needle/haystack theorem, and one might consider only multiples of (say) 24 or 72 after a certain point, noticing that multiple factors of 2 and 3 were prevalent.

A better approach is to realize that the sum of factors for a number of the form $2^a \cdot 3^b \cdot 5^c \cdot \ldots \cdot p^n$ is equal to $(2^{a+1} - 1)/(2 - 1) \cdot (3^{b+1} - 1)/(3 - 1) \cdot \ldots \cdot (p^{n+1} - 1)/(p - 1)$. First of all, this allows one to recognize that he or she is trying to generate a product of factors that multiplies to 3.5, 4.5, or 5.5 of the form (after rearranging terms a little)

$$\frac{2}{(2-1)} \cdot \left[1 - 1/(2^{a+1})\right] \cdot \frac{3}{(3-2)} \cdot \left[1 - 1/(3^{b+1})\right] \cdot \ldots \cdot \frac{p}{(p-1)} \cdot \left[1 - 1/(p^{n+1})\right].$$

This immediately demonstrates, for example, that a number with a multiplicity of 3.5 must contain at least 3 different factors, because the maximum product possible from 2 factors is $2/(2 - 1) \cdot 3/(3 - 1) = 3$. More importantly, it allows additional insights – for example, that a number of multiplicity 3.5 which only contains a single factor of 2 must have at least 5 different prime factors because the maximum possible multiplicity for a number with a single factor of 2 and only 3 other prime factors is $3/2 \cdot 3/(3 - 1) \cdot 5/(5 - 1) \cdot 7/(7 - 1) = 105/32 = 3.28125 < 3.5$.

Furthermore, a number with a single factor of 2 which is multiplicity 3.5 must have at least one factor of 3, because if $7/2 = 3/2 \cdot \text{“stuff”}$, then “stuff” must have at least one factor of 3 in the denominator, which can only come from a power of 3.

Here four approaches, each more sophisticated than the last:

(1) Try integers until you hit the ones you are looking for. If you have a fast computer and enough time, you can do this. It is worth noting that all answers were multiples of 24.

(2) Try creating loops on a finite but reasonable set of factors (of note is that among the 11 numbers above, the following prime powers are represented:

a. $2, 3, 5, 7, 8, 9, 10$  

b. $3, 2, 4$  

c. $5, 1, 2$  

d. $7, 1, 2$  

e. $11, 0, 1$  

f. $13, 0, 1$  

g. $17, 0, 1$

h. $19, 0, 1$ – note that $1 + 7 + 19 = 3 \times 19$

i. $23, 0, 1$ – note that $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 = 23 \times 89$
work effectively in problems with higher bounds or multiplicities.

If we add these together, we get $(-5 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 7 \times 73$; hence, using exactly 8 powers of 2 leads to using at least one factor of 73, and using a single factor of 73 leads to using a factor of 37 (this observation can be useful in approach (4), below).

Additionally, note that factors of 13 are generated both from using $2^1 (4095 = 3^2 \times 5 \times 7 \times 13)$ and from using $3^2 (13 = 1 + 3 + 5^2)$, and factors of 31 are generated both from using $2^9 (1023 = 3 \times 11 \times 31)$ and from using $5^2 (31 = 1 + 5 + 5^2)$.

While it’s certainly far from obvious to restrict a search to precisely these primes and powers, this sort of search would be much faster than trying, say, every 72nd value between 0 and $10^{15}$. The total number of possibilities in this list is clearly less than 400,000. Expanding beyond this list, of course, carries a cost...

(3) Consider an exhaustive branch-and-bound search – consider all possible powers of 2 up to the highest possible value, and then consider what additional primes would be required to push the factor above the threshold. For example, when searching for a number of multiplicity 5.5, consider what happens if you use $2^1$: You get 3/2. Then if you incorporate at least one factor of each prime up to 47, the multiplicity is still bounded above by \( (3/2) \times (3/(3 - 1)) \times (5/(5 - 1)) \times (7/(7 - 1)) \times \ldots \times (47/(47 - 1)) = 5.407 \ldots \), which is less than 5.5. Thus, a number with a single factor of 2 cannot be have multiplicity 5.5 unless it has at least 15 additional prime factors. But the smallest number with 16 prime factors is more than 30,000 times too big. This is a very easy way to show that a number of multiplicity 5.5 that is less than $10^{15}$ must have at least 2 factors of 2; a similar argument shows that it can have at most 31 factors of 2 (no matter how many factors of 2 a number has, its multiplicity is less than 5.5 if it doesn’t include at least 6 other prime factor...) A systematic approach like this covering the first few factors, along with a “hunt-and-peck” looping approach as in (2) above, should quickly find all possible answers for the problem as stated.

(4) A more advanced approach is to think of prime powers as integer combinations of vectors, where the vectors are based on the number of powers of each prime factor. For example, 4320 = $2^5 \times 3^3 \times 5$. Note that if we re-express each of these in a vector based on the powers of 2, 3, 5, and 7 that appear in the fraction representing the multiplicity of that factor, we get

- $2^5$ has a multiplicity factor of $63/32 = 2^{-5} \times 3^2 \times 5^0 \times 7^1$, or $(-5, 2, 0, 1)$
- $3^3$ has a multiplicity factor of $40/27 = 2^3 \times 3^{-3} \times 5^1 \times 7^0$, or $(3, -3, 1, 0)$, and
- $5^1$ has a multiplicity factor of $6/5 = 2^1 \times 3^1 \times 5^{-1} \times 7^0$, or $(1, 1, -1, 0)$.

If we add these together, we get $(-5 + 3 + 2 - 3 + 1, 0 + 1 - 1, 1 + 0 + 0) = (-1, 0, 0, 1)$ which represents the number $2^{-1} \times 3^0 \times 5^0 \times 7^1 = 7/2 = 3.5$. Combining an approach such as this with (3) and (2) above is probably more powerful and work than is necessary for this problem, but might work effectively in problems with higher bounds or multiplicities.