Disclaimer: These are my personal notes taken in a student-run seminar that ran Fall 2018 at IUPUI. I do not claim completeness nor originality of the work. All of the results can be found in at least one of the sources referenced below. If there is anything regarding these notes that you would like me to learn (mistakes, typos, interesting facts, historical context, etc.), please contact me at abarhoum@iupui.edu.
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Chapter 1

The Nevanlinna Characteristic

Suppose \( a, b \) are two distinct values that a function meromorphic in \( \mathbb{C} \), say \( f \), takes. Is there a sense that the number of times \( f \) takes the value \( a \) in \( D_r(0) \) is asymptotically the same as the number of times the value \( b \) is achieved in \( D_r(0) \)?

1.1 Jensen’s Formula

We need to define a function that does the counting.

**Definition 1.1.1.** For \( a \in \hat{\mathbb{C}} \) and \( f \) meromorphic (non-constant) in \( \mathbb{C} \), \( n_f(r,a) \) is the number of times \( f \) takes the value \( a \) in \( D_r(0) \).

If \( f(z_j) = a \) for \( \{z_j\}_{j=1}^{V_r} \) (\( V_r \) is necessarily finite) with multiplicity \( m_j \), then

\[
  n_f(r,a) = \sum_{j=1}^{V_r} m_j.
\]

It turns out that a different counting function appears when you study Jensen’s formula. Recall that for an entire \( f(z) \), \( f(0) \neq 0 \), and for any \( r > 0 \),

\[
  \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \ d\theta = \log |f(0)| + \sum_{j=1}^{n_f(r,0)} \log \left( \frac{r}{|z_j|} \right) \tag{1.1}
\]

where \( \{z_j\}_{j=1}^{n_f(r,0)} \) are the zeros of \( f \) in \( D_r(0) \), counting multiplicity. A proof can be found in [5, chapter 9]. We would like to extend this formula to include meromorphic functions and to allow zeros/poles at \( z = 0 \). If we start with \( f(z) \) meromorphic in \( D_r(0) \), \( f(0) \neq 0, \infty \), then it possesses finitely many poles the location of which we label \( \{w_1\}_{i=1}^k \) (\( w_i \)'s repeat according to multiplicity) and so, the function \( g(z) := f(z)(z - w_1) \cdots (z - w_k) \) is entire and one can apply
(1.1) to get
\[
\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| \, d\theta = \log|f(0)| + \sum_{j=1}^{n_f(r,0)} \log \left( \frac{r}{|z_j|} \right) - \sum_{j=1}^{n_f(r,\infty)} \log \left( \frac{r}{|w_j|} \right). \tag{1.2}
\]
Next, if \( f(0) = 0, \infty \), then write
\[
f(z) = \eta(f) z^{m_0} \left(1 + O(z)\right), \quad \eta(f) \neq 0
\]
and apply (1.2) to \( g(z) := f(z)/z^{m_0} \) to yield
\[
\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| \, d\theta
= \log|\eta(f)| + \sum_{j=1}^{n_f(r,0)} \log \left( \frac{r}{|z_j|} \right) - \sum_{j=1}^{n_f(r,\infty)} \log \left( \frac{r}{|w_j|} \right) + m_0 \log r \tag{1.3}
\]
where \( m_0 \) can be written as \( n_f(0,0) - n_f(0,\infty) \)

**Definition 1.1.2.** The counting function, \( N_f(r,a) \), is given by
\[
N_f(r,a) := \int_0^r \frac{n_f(s,a) - n_f(0,a)}{s} \, ds + n_f(0,a) \log r
\]
Observe that for \( \{z_j\}_{j=1}^m \) such that \( f(z_j) = a \) with multiplicity \( m_j \), we have that
\[
N_f(r,a) = \int_0^r \frac{n_f(s,a) - n_f(0,a)}{s} \, ds + n_f(0,a) \log r
= \sum_{z_j \neq 0} m_j \log \left( \frac{r}{|z_j|} \right) + n_f(0,a) \log r. \tag{1.4}
\]
Hence, we have proven the following theorem

**Theorem 1.1.3.** For \( f(z) \) meromorphic in \( \mathbb{C} \) and \( \forall r > 0 \)
\[
\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| \, d\theta = \log|\eta(f)| + N_f(r,0) - N_f(r,\infty). \tag{1.5}
\]

**Example 1.1.3.1.** Let \( f(z) = e^z \) and consider \( a \neq 0, \infty \). Then \( N_f(r,0) = 0 \) while \( N_f(r,a) \sim r/\pi \) \((z = \log a \pm 2\pi in, \ n \in \mathbb{Z})\).

### 1.2 The First Main Theorem

The last example showed us that \( N_f(r,a) \) is highly dependent on \( a \), and so needs to be complemented by some other term.

Turns out, the correct ingredient is the following. With \( u_+ = \max\{0, u\} \),
1.2. THE FIRST MAIN THEOREM

**Definition 1.2.1.** The Proximity Function if given by

\[ m_f(r, \infty) := \frac{1}{2\pi} \int_0^{2\pi} \log_+ |f(re^{i\theta})| \, d\theta, \]

\[ m_f(r, a) := m_{f-a}^{-1}(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log_+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \, d\theta. \]

As the name suggests, \( m_f(r, a) \) measures the average proximity of \( f(z) \) to \( a \) on the circle \( |z| = r \). With this notation and using the identity \( \max\{0, x\} + \max\{0, -x\} = |x| \), note that theorem 1.1.3 becomes

\[ m_f(r, \infty) - m_f(r, 0) = \log |\eta(f)| + N_f(r, 0) - N_f(r, \infty). \] (1.6)

Observe that this implies that the expression \( m_f(r, a) + N_f(r, a) \) changed by a number independent of \( r \) when for \( a = 0, \infty \). One is led to ask whether or not this is true for all \( a \).

**Lemma 1.2.2.** For \( x, y > 0 \),

(a) \( \log_+ (x + y) \leq \log_+ (x) + \log_+ (y) + \log 2 \),

(b) \( \log_+ (|x - y|) \leq \log_+ (x) + \log_+ (y) \)

(c) \( \log_+ (xy) \leq \log_+ (x) + \log_+ (y) \)

**Proof.** WLOG suppose \( 0 < y \leq x \). Then by the monotonicity of \( \log_+ (\cdot) \), then part (a), (c) can be verified by the identity \( (u + v)_+ \leq u_+ + v_+ \):

\[ \log_+ (x + y) \leq \log_+ (2x) = (\log(2) + \log(x))_+ \leq \log_+ (x) + \log_+ (y) + \log 2 \]

\[ \log_+ (xy) = (\log(x) + \log(y))_+ \leq \log_+ (x) + \log_+ (y) \]

while the second inequality is easier and follows from monotonicity and \( |x - y| \leq |x| = x \). \( \Box \)

Now, apply the identity (1.6) to the function \( f-a \) and observe that \( N_{f-a}(r, 0) = N_f(r, a) \), \( N_{f-a}(r, \infty) = N_f(r, \infty) \), and \( m_{f-a}(r, 0) = m_f(r, a) \). This yields the identity

\[ m_{f-a}(r, \infty) + N_f(r, \infty) = \log |\eta(f - a)| + N_f(r, a) + m_f(r, a). \] (1.7)

Finally, with the help of lemma 1.2.2,

\[ |m_{f-a}(r, \infty) - m_f(r, \infty)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\log_+ |f(z) - a| - \log_+ |f(z)|| \, d\theta \leq \log_+ |a| + \log 2. \] (1.8)

We have now shown the following

**Theorem 1.2.3** (First Main Theorem). For any \( a \in \hat{C} \)

\[ N_f(r, a) + m_f(r, a) = N_f(r, \infty) + m_f(r, \infty) + O(1). \] (1.9)

Furthermore, the \( O(1) \) term is uniformly bounded above by \( |\log |\eta(f - a)|| + \log_+ |a| + \log 2 \).
CHAPTER 1. THE NEVANLINNA CHARACTERISTIC

This allows us to make a new definition:

Definition 1.2.4. For a function \( f \) meromorphic in \( \mathbb{C} \), the Nevanlinna Characteristic, \( T_f(r) \) (sometimes \( T(r,f) \)), is given by

\[
T_f(r) := N_f(r, \infty) + m_f(r, \infty) \tag{1.10}
\]

Example 1.2.4.1. Let \( f(z) = e^z \). Then, \( \log |f(re^{i\theta})| = r \cos \theta \) and

\[
m_f(r, \infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log_+ |f(re^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} r \cos \theta \, d\theta = \frac{r}{\pi}.
\]

and since \( N_f(r, \infty) = 0 \), we have that \( T_f(r) = r/\pi \). For an application of this information, set \( a = 1 \). Then

\[
N_f(2\pi m, 1) = n_f(0, 1) \log(2\pi m) + 2 \sum_{j=1}^{m-1} \log \left( \frac{2\pi m}{|2\pi j|} \right) = \log(2\pi m) + 2(m - 1) \log m - 2 \log((m - 1)!) = \log(2\pi m) + 2m \log m - 2 \log m!.
\]

Furthermore,

\[
m_f(2\pi m, 1) = \frac{1}{2\pi} \int_0^{2\pi} \log_+ \left| \frac{1}{e^{2\pi m \cos \theta} - 1} \right| \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log_+ \left( \frac{1}{e^{2\pi m \cos \theta} - 1} \right) \, d\theta = \frac{1}{\pi} \int_0^{2\pi} \log_+ \left( \frac{1}{e^{2\pi m \cos \theta} - 1} \right) \, d\theta
\]

where \( \xi = \cos^{-1} \left( \frac{\log 2}{2\pi m} \right) < \pi/2 \forall m > 1 \).

\[
= \frac{1}{\pi} \int_{\xi}^{\pi/2} \log_+ \left| \frac{1}{e^{2\pi m \cos \theta} - 1} \right| \, d\theta \leq \frac{1}{\sqrt{2\pi m \cos \theta}}
\]

Next, note that

\[
\log \left| \frac{1}{1 - e^{2\pi m \cos \theta}} \right| \leq \frac{1}{\sqrt{2\pi m \cos \theta}}
\]

the later of which is integrable for all \( m > 1 \). Hence, by dominated convergence theorem we deduce that \( m_f(2\pi m, 1) = o(1) \). Plugging this into (1.9) yields

\[
\log(m!) = \left( m + \frac{1}{2} \right) \log m - m + \log 2\pi + O(1)
\]

This is, up to a bounded term, Stirling’s Approximation!

1.3 Properties of the Nevanlinna Characteristic

Proposition 1.3.1. For \( f, g \) meromorphic in \( \mathbb{C} \) and \( \lambda \in \mathbb{C} \setminus \{0\} \),

(a) \( T_f(\lambda f) = \lambda T_f(r) \)

\( \)
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(b) $T_{\lambda_f}(r) \leq T_f(r) + O(1)$
(c) $T_{fg}(r) \leq T_f(r) + T_g(r)$
(d) $T_{f+g}(r) \leq T_f(r) + T_g(r) + O(1)$
(e) $T_{1/f}(r) = T_f(r) + O(1)$

Proof. Part (a) is easy since $n_{f}(r, \infty) = n_{f}(r, \infty)$, $n_{f}(r, 0) = n_{f}(r, 0)$ and $\log_{+}|f^n| = n \cdot \log_{+}|f|$.

Part (b) follows from the fact that $N_{f+g}(r, \infty) = n_{f}(r, \infty)$, $n_{f}(r, 0) = n_{f}(r, 0)$ and

$$\log_{+}|\lambda_f| \leq \log_{+}|\lambda| + \log_{+}|f|.$$ 

Part (c) follows from the fact that

$$N_{fg}(r, \infty) \leq N_f(r, \infty) + N_g(r, \infty)$$

which can be deduced from the second expression of (1.4) by noting that the order of poles/zeros of $fg$ is smaller than the sum of orders of $f$ and $g$ due to possible cancellations. An application of part (c) of Lemma (1.2.2) yields a similar inequality for the proximity function. Part (d) of the present proposition can be proven similarly.

Finally, since $N_f(r, 0) = N_{1/f}(r, \infty)$, $m_f(r, 0) = m_{1/f}(r, \infty)$ and by (1.6), part (e) follows. \hfill \Box

In the case of $f(z) = (P/Q)(z)$, $P, Q$ polynomials, the Nevanlinna characteristic becomes easy to right down. Suppose $\deg P = l, \deg Q = m$, then $f(z) = Cz^{l-m} + O(z^{l-m-1})$. This implies that $m_f(r, \infty) = \max\{l - m, 0\} \cdot \log(r) + O(1)$. Furthermore, for $r$ large enough, $n_f(r, \infty) = m \implies N_f(r, \infty) = m \log(r) + O(1)$. Finally, observe that $m + \max\{l - m, 0\} = \max\{m, l\} = \max\{\deg P, \deg Q\} := d$ is the degree of a rational function. Hence, we have shown that

**Theorem 1.3.2.** Let $f(z) \equiv (P/Q)(z)$ be a rational function with degree $d = \max\{\deg P, \deg Q\}$, then

$$T_f(r) = d \log r + O(1). \tag{1.11}$$

It turns out, this growth rate is characteristic of rational functions, i.e. for $f(z)$ meromorphic but not rational, we will prove that $\lim_{r \to \infty} \frac{T_f(r)}{\log r} \to \infty$.

**Lemma 1.3.3.** For an entire function $f$, define $M_f(r) := \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|$. Then

$$T_f(r) \leq \log_{+} M_f(r) \leq 3T_f(2r)$$

Proof. The first inequality is easy since $\log_{+}(z)$ is an increasing function and

$$T_f(r) = N_f(r, \infty) + \int_0^{2\pi} \log_{+} |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \log_{+} M_f(r) \cdot \int_0^{2\pi} \frac{d\theta}{2\pi} = \log_{+} M_f(r).$$

As for the second inequality, fix $r$ and consider $D_{2r}(0)$. Since $f(z)$ is entire, it admits the factorization $f(z) = g(z) \cdot B(z)$ where $g(z)$ is non-vanishing and $B(z)$ is a Blaschke-type product with argument scaled by $2r$ so that $|B(z)| < 1$ for
z \in \mathbb{D}_{2r}(0) and |B(z)| = 1 on \partial \mathbb{D}_{2r}(0). Then, \log |g(z)| is a harmonic function on \mathbb{D}_{2r}(0), and while \( g(z) \) may have a zero on \( \partial \mathbb{D}_{2r}(0) \), \( \log |g(z)| \) is still integrable. This allows us to apply Poisson’s formula, namely for \( z \in \mathbb{D}_{2r}(0) \),

\[
\log |f(z)| \leq \log |g(z)|
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} P_{1/2}(\phi - \theta) \log |g(2re^{i\theta})| \, d\theta \quad \text{where} \quad P_{1/2}(\theta) := \frac{1 - r^2}{1 - 2r \cos \theta + r^2}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} P_{1/2}(\phi - \theta) \log |f(2re^{i\theta})| \, d\theta \leq 3 \cdot \frac{1}{2\pi} \int_0^{2\pi} \log_+ |f(2re^{i\theta})| \, d\theta = 3T(2r)
\]

where we use the fact that \( P_{1/2}(\theta) \) is positive for \( r < 1 \) and \( P_{1/2}(\theta) \leq 3 \) for \( r = 1/2 \).

**Theorem 1.3.4.** If \( f \) is not a rational, meromorphic function, then

\[
\frac{T_f(r)}{\log r} \to \infty \quad \text{as} \quad r \to \infty.
\]

**Proof.** Suppose to the contrary that \( T_f/\log r \to b \neq \infty \) as \( r \to \infty \)

Case 1: Suppose \( f(z) \) is entire. Then

\[
\log_+ M_f(r) \leq 3T_f(2r) \implies M_f(r) \leq r^c
\]

for some constant \( c \), which (by Liouville’s theorem) implies \( f(z) \) is polynomial, a contradiction.

Case 2: \( f(z) \) has finitely many poles, then there is \( p(z) \) polynomial such that \( g(z) = f(z) \cdot p(z) \) is entire. By case 1, \( g(z) \) is polynomial and \( f(z) \) is rational, a contradiction.

Case 3: \( f(z) \) has infinitely many poles. Then

\[
N_f(r, \infty) \geq \int_{\sqrt{r}}^r \frac{n_f(s, \infty) - n_f(0, \infty)}{s} \, ds \geq \frac{1}{2} n_f(\sqrt{r}, \infty) \log(r) - \frac{1}{2} n_f(0, \infty) \log r
\]

which yields the inequality

\[
\frac{N_f(r, \infty)}{\log r} \geq \frac{1}{2} n_f(\sqrt{r}, \infty) - \frac{1}{2} n_f(0, \infty)
\]

and so

\[
\liminf_{r \to \infty} \frac{N_f(r, \infty)}{\log r} \geq \frac{1}{2} n_f(\sqrt{r}, \infty) - \frac{1}{2} n_f(0, \infty)
\]

and since \( \lim_{r \to \infty} n_f(\sqrt{r}, \infty) = \infty \) by hypothesis, and by the positivity of \( m_f(r, \infty) \), the result follows.

In particular, for any non-constant function meromorphic in \( \mathbb{C} \), we have that

\[
\lim_{r \to \infty} T_f(r) = \infty \quad (1.12)
\]

An immediate consequence is the following
**Corollary 1.3.5.** For any non-constant function meromorphic in $\mathbb{C}$ and any $a \in \hat{\mathbb{C}}$,

$$\limsup \frac{N_f(r,a)}{T_f(r)} \leq 1$$

**Proof.** By the FMT,

$$\frac{N_f(r,a)}{T_f(r)} = \frac{T_f(r) + O(1) - m_f(r,a)}{T_f(r)} \leq 1 + \frac{O(1)}{T_f(r)}$$

by the positivity of $m_f(r,a)$, $T_f(r,a)$ and the result follows by (1.12). \qed

**Remark 1.3.6.** In fact, the limit

$$\lim_{r \to \infty} \frac{N_f(r,a)}{T_f(r)} = 1$$

for all $a \in \mathbb{C} \setminus E$ where $E$ is a set of (inner) capacity 0. This fact is shown in [4, page 276] and will not be discussed in these notes.

### 1.4 Valiron’s Lemma

Observe that for $S(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ (Möbius Transformation), the statement

$$T_{S \circ f}(r) = T_f(r) + O(1)$$

is equivalent to the FMT. This can be shown with the help of proposition (1.3.1),

$$T_{a_{f+b}/c} (r) = T_{a/c} - T_{1/(af+db)} (r) + O(1) \leq T_{ef+d(r)} + O(1) \leq T_f(r) + O(1).$$

On the other hand,

$$T_{af/a}(r) \leq T_{af(r)} + O(1) \leq T_{af+b}(r) + O(1) = T_{/af+b(r)}(r) + O(1) \leq T_{c+(ad-bc)/(af+db)}(r) + O(1) = T_{e+d(r)} + O(1) = T_{ef+d}(r) + O(1).$$

and the conclusion follows. This is a special case of a more general theorem

**Lemma 1.4.1** (Valiron’s Lemma). Let $f$ be any non-constant meromorphic function and $R$ any rational function of degree $d \geq 1$. Then $F = R \circ f$ has characteristic

$$T_F(r) = dT_f(r) + O(1) \quad \text{as} \quad r \to \infty$$

**Proof.** Let $R = P/Q$ where $P, Q$ are polynomials and WLOG suppose $d := \deg Q \geq \deg P$ (otherwise consider $1/R$ and apply the FMT). Let $p := P \circ f$, $q := Q \circ f$. Let $E_\epsilon = \{z \mid |q(z)| < \epsilon\}$. On such sets, since $p, q$ do not share zeros we can find a $C$ such that $1/C < |p(z)| < C$, while on the complement (since $\deg P \leq \deg Q$) we have that $|F(z)|, |1/q(z)|$ are bounded. Hence, (remember that $(u+v)_+ \leq u_+ + v_+$)

$$\log_+ |F(z)| = \log_+ |1/q(z)| + O(1) \quad \forall z \in \mathbb{C}$$
This implies \( m_F(r) = m_{1/q}(r) + O(1) \). Since \( q, F \) share poles, \( N_F(r) = N_{1/q}(r) \). Hence, \( T_F(r) = T_{1/q}(r) + O(1) = T_q(r) + O(1) \).

At this point, it suffices to show that for any polynomial \( Q(z) \), \( \deg Q = d \) we have \( T_q(r) = dT_f(r) + O(1) \) (where \( q = Q \circ f \) as in above).

Indeed, one can immediately conclude \( N_q(r) = dN_f(r) \) since multiplicity of poles is multiplied by \( d \). As for \( m_q(r) \), write \( Q(z) = a_dz^d + \cdots + a_0 \) note that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log+ |a_d f^d + \cdots + a_0| \, d\theta = \left( \int_{|f| > R} + \int_{|f| < R} \right) \log+ |a_d f^d + \cdots + a_0| \, d\theta
\]

\[
= \int_{|f| > R} \log+ |a_d f^d + \cdots + a_0| \, d\theta + O(1)
\]

Where \( R \) is chosen such that \( \forall z \) such that \( |z| > R \), \( |Q(z)| > m > 1 \). Then, \( \log+ (|Q(f)|) = \log(|Q(f)|) \) and

\[
\int_{|f| > R} \log+ |a_d f^d + \cdots + a_0| \, d\theta - \int_{|f| > R} \log+ |f^d| \, d\theta
\]

\[
= \int_{|f| > R} \log \left| a_d + \frac{a_d-1}{f} + \cdots + \frac{a_0}{f^d} \right| \, d\theta = O(1)
\]

and we now have that \( m_q(r) = d \cdot m_f(r) + O(1) \implies T_q(r) = dT_f(r) + O(1) \).

Applying this to our current problem yields

\[
T_F(r) = dT_f(r) + O(1)
\]

\[\square\]

The general form of Valiron’s lemma is due to Mokhon’ko and states

**Theorem 1.4.2.** Let \( f \) be a non-constant meromorphic function and consider

\[
F(z) = \frac{P_m(z)}{Q_n(z)} = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_m(z)f^n(z)}{b_0(z) + b_1(z)f(z) + \cdots + b_n(z)f^n(z)}, \quad b_m, a_n \neq 0
\]  

(1.13)

where \( a_j(z), b_j(z) \) are meromorphic, \( P_m(z), Q_n(z) = 1 \) (i.e. \( P_m(z, w), Q_n(z, w) \) have no common factors over the field of rational functions). Then

\[
T(r, F) = dT(r, f) + O\left( \sum_{i=1}^{m} T(r, a_i) + \sum_{i=1}^{n} T(r, b_i) \right)
\]

(1.14)

where \( d = \max\{m, n\} \)

Note that we have adopted the notation \( T_f(r) \equiv T(r, f) \) for convenience. The main tools in this proof will be proposition 1.3.1, which implies that for \( R(z_1, z_2, \ldots, z_q) \) rational,

\[
T(r, R(f_1, \ldots, f_q)) \leq O\left( \sum_{j=1}^{q} T(r, f_j) \right)
\]

(1.15)

as well as the following two lemmas.
Lemma 1.4.3. Let $\{\varphi_k\}, f$ be meromorphic functions in $\mathbb{C}$ and let

$$A(z) = \left( \varphi_1 f + \varphi_2 f^2 + \cdots + f^m \right) \cdot f^{m-2} \quad (1.16)$$

then, there exist $\{u_{s}\}_{s=0}^{m-1}, \{q_{s}\}_{s=0}^{m-2}$ meromorphic in $\mathbb{C}$ such that $u_{m-1} \equiv 1$ and

$$A(z) + \sum_{s=0}^{m-2} q_{s} f^{s} = \left[ F_{m-1}(z) \right]^{2} \quad \text{where} \quad F_{m-1}(z) = u_{m-1} f^{m-1} + \cdots + u_{0}. \quad (1.17)$$

Moreover,

$$q_{k} = \sum_{s+t=k, \ s,t \leq m-1} u_{s} u_{t}, \ k = 0, ..., m-2 \quad \text{and}$$

and $u_{j}$'s are polynomials with constant coefficients in $\{\varphi_i\}_{i=1}^{m-1}$.

Proof. We start with the right hand side and construct the set $\{u_{s}\}$ inductively. Suppose we have $u_{m-1}, ..., u_{p}, u_{p-1} \equiv 1$ and $1 \leq p < m-1$. By definition of $F_{m-1}$,

$$[F_{m-1}]^{2} = \sum_{k=0}^{m-2} f^{k} \left( \sum_{s+t=k, \ s,t \leq m-1} u_{s} u_{t} \right) \equiv \sum_{k=0}^{m-2} f^{k} q_{k} + \sum_{k=m-1}^{2m-2} f^{k} q_{k}$$

$$= \sum_{k=0}^{m-2} f^{k} q_{k} + f^{m-2} \sum_{k=1}^{m} f^{k} q_{k+m-2}$$

where the last step follows by re-indexing the sum. The goal now is, given $u_{m-1}, ..., u_{p},$ to choose $u_{p-1}, ..., u_{0}$ such that the first sum in the last equality is $A(z).$ We begin by solving for $u_{p-1}.$ Observe that we would like

$$q_{m+p-2} = \varphi_{p} = \sum_{s+t=m+p-2, \ s,t \leq m-1} u_{s} u_{t} = 2 u_{m-1} u_{p-1} + \sum_{s+t=m+p-2, \ s,t \leq m-1} u_{s} u_{t} \quad (1.18)$$

and observe that the conditions $s+t = m+p-2, \ s,t \leq m-1, \text{ and } p < m-1 \implies s \geq p, t \geq p,$ and so we can solve for $u_{p-1}$ in terms of $u_{p}, ..., u_{m-1}$ since

$$u_{p-1} = \frac{1}{2} \left( \varphi_{p} - \sum_{s+t=m+p-2, \ s,t \leq m-2} u_{s} u_{t} \right).$$

Continuing in this fashion, we can find all required $u_{j}$'s, which proves the lemma.\[\square\]

Next, we observe a weaker version of the polynomial case of the Mokhon’ko’s theorem.
Lemma 1.4.4. Let
\[ F_m(z) = a_0(z) + a_1(z)f(z) + \cdots + a_m(z)f^m(z), \] then
\[ T(r, F_m) \leq mT(r, f) + O \left( \sum_{i=0}^{m} T(r, a_i) \right) \] (1.19)

Proof. The case \( m = 0 \) is obvious. From here we proceed by induction. Suppose we have the result for \( m = n - 1 \), then by writing \( F_n = a_0 + f \cdot F_{n-1} \), it follows from proposition 1.3.1 and the induction hypothesis that
\[ T(r, F_n) \leq T(r, a_0) + T(r, f) + T(r, F_{n-1}) + O(1) = nT(r, f) + O \left( \sum_{i=1}^{n} T(r, a_i) \right) \] (1.20)

We are now ready to prove the main theorem

Proof of theorem 1.4.2. We consider two separate cases.

Case 1: \( Q_n(z) \equiv 1 \). Then \( d = m \) and \( F = a_0 + \cdots + a_m f^m \). By lemma 1.4.4, we already have that
\[ T(r, F) \leq mT(r, f) + O \left( \sum_{i=1}^{m} T(r, a_i) \right). \]

To prove the opposite inequality, we proceed by induction. For a base case, let \( m = 1 \) \( \implies F = a_0 + a_1 f \implies F = (f - a_0)/a_1 \) and
\[ T(r, f) = T \left( r, \frac{F - a_0}{a_1} \right) \leq T(r, F) + O \left( T(r, a_0) + T(r, a_1) \right). \]

Next, suppose we have \( T(r, F) \geq mT(r, f) + O \left( \sum_{i=1}^{n} T(r, a_i) \right) \) for \( m = n - 1 \) and let’s prove this result for \( m = n \). Observe that
\[ \left( \frac{F - a_0}{a_m} \right) f^{m-2} = \left( \frac{a_1}{a_m} f + \cdots + f^m \right) f^{m-2} \]
and by lemma 1.4.3, we can write
\[ L(z) := \left( \frac{F - a_0}{a_m} \right) f^{m-2} + q_{m-2} f^{m-2} + \cdots + q_0 = [\tilde{F}_{m-1}]^2 \]
where \( \tilde{F}_{m-1} \) is a polynomial in \( f \) of degree \( m - 1 \) with coefficients meromorphic in \( \mathbb{C} \). On the one hand, by the induction hypothesis and lemma 1.4.4,
\[ T(r, L(z)) = T(r, \tilde{F}_{m-1}^2) = 2T(r, \tilde{F}_{m-1}) = 2(m - 1)T(r, f) + O \left( \sum_{i=1}^{n} T(r, a_i) \right). \] (1.21)
On the other hand, applying the induction hypothesis and (1.20) of lemma 1.4.4 to the left hand side yields

$$T(r, L(z)) \leq (m - 2)T(r, f) + T\left( r, \frac{F - a_0}{a_m} + q_{m-2} \right) + O\left( \sum_{i=1}^{m-3} T(r, q_i) \right)$$

Since $q_i$’s are polynomials in $\frac{a_i}{a_m}$ with constant coefficients,

$$O\left( \sum_{i=1}^{m-3} T(r, q_i) \right) = O\left( \sum_{i=1}^{m} T(r, a_i) \right),$$

and hence,

$$T(r, L(z)) \leq (m - 2)T(r, f) + T(r, F) + O\left( \sum_{i=1}^{m} T(r, a_i) \right).$$

Using this and (1.21) yields

$$2(m - 1)T(r, f) + O\left( \sum_{i=1}^{m} T(r, a_i) \right) \leq (m - 2)T(r, f) + T(r, F) + O\left( \sum_{i=1}^{m} T(r, a_i) \right)$$

(1.22)

$$mT(r, f) + O\left( \sum_{i=1}^{m} T(r, a_i) \right) \leq T(r, F)$$

(1.23)

This finishes the proof for case 1.

Case 2: $Q_n(z) \not\equiv 1$. WLOG, we suppose that $m \geq n$ (otherwise, apply $T(r, f) = T(r, 1/f) + O(1)$). Note that Case 1 was $m > n = 0$. We know in the polynomial case that

$$T(r, F) \leq mT(r, f) + O\left( \sum_{i=1}^{m} T(r, a_i) \right).$$

We proceed by induction on the degree of $Q_n(z)$. Suppose that for any $F$ such that $\deg Q_n \leq k - 1$, $\deg P_m = m$, and $m \geq k - 1$ we have the above result. We would like to prove the above inequality for the case where $\deg Q_n = k$, $\deg P_m = m$, $m \geq k$. To this end, observe that by long division,

$$F = \frac{P_m}{Q_n} = c_m f^{m-k} + \cdots + c_0 + \frac{\tilde{b}_{k-1} f^{k-1} + \cdots + \tilde{b}_0}{Q_n}$$
then, we have that
\[
T(r, F) \leq T(r, c_{m-k} f^{m-k} + \ldots + c_0) + T \left( r, \frac{b_{k-1} f^{k-1} + \ldots + b_0}{Q_n} \right) \quad (1.24)
\]
\[
T(r, F) \leq T(r, c_{m-k} f^{m-k} + \ldots + c_0) + T \left( r, \frac{Q_n}{b_{k-1} f^{k-1} + \ldots + b_0} \right) \quad (1.25)
\]
\[
= (m-k) T(r, f) + k T(r, f) + O \left( \sum_{i=0}^{m} T(r, a_i) + \sum_{i=0}^{k} T(r, b_i) \right) \quad (1.26)
\]
\[
= m T(r, f) + O \left( \sum_{i=1}^{m} T(r, a_i) + \sum_{i=1}^{k} T(r, b_i) \right) \quad (1.27)
\]
where the last equality follows from Case 1 and the induction hypothesis. This proves the inequality in general. To arrive at the opposite inequality, note that
\[
(P_m, Q_n) = 1 \implies \exists U_s(z, w) = c_0 + \cdots + c_s w^s, V_t(z, w) = \tilde{c}_0 + \cdots + \tilde{c}_t w^t
\]
where \(c_j, \tilde{c}_j\)’s are polynomials in \(a_i, b_i\)’s with constant coefficients and \(t > s\) such that
\[
P_m U_s + Q_n V_t = 1 \implies \frac{Q_n}{P_m} + \frac{U_s}{V_t} = \frac{1}{P_m V_t}
\]
and so, on the one hand,
\[
T \left( r, \frac{Q_n}{P_m} + \frac{U_s}{V_t} \right) = T \left( r, \frac{1}{P_m V_t} \right)
\]
\[
= T(r, P_m V_t) = (m+t) T(r, f) + O \left( \sum_{i=0}^{n} T(r, a_i) + \sum_{i=0}^{m} T(r, b_i) \right)
\]
while on the other hand, we have
\[
T \left( r, \frac{Q_n}{P_m} + \frac{U_s}{V_t} \right) \leq T \left( r, \frac{Q_n}{P_m} \right) + T \left( r, \frac{U_s}{V_t} \right) + O(1)
\]
\[
\leq T \left( r, \frac{Q_n}{P_m} \right) + t T(r, f) + O \left( \sum_{i=0}^{n} T(r, a_i) + \sum_{i=0}^{m} T(r, b_i) \right)
\]
This gives
\[
(m+t) T(r, f) + O \left( \sum_{i=0}^{n} T(r, a_i) + \sum_{i=0}^{m} T(r, b_i) \right) \leq
\]
\[
T \left( r, \frac{Q_n}{P_m} \right) + t T(r, f) + O \left( \sum_{i=0}^{n} T(r, a_i) + \sum_{i=0}^{m} T(r, b_i) \right) \quad (1.28)
\]
and hence

\[ mT(r, f) + O \left( \sum_{i=0}^{n} T(r, a_i) + \sum_{i=0}^{m} T(r, b_i) \right) \leq T \left( r, \frac{Q_n}{P_m} \right) = T \left( r, \frac{P_m}{Q_n} \right) \] (1.29)

\[ \square \]

**Remark 1.4.5.** Note that the above proof was purely algebraic and relied entirely on proposition 1.3.1 as well as \( T(r, f) = T(r, 1/f) + O(1) \).

The essentialness of the \( O(1) \) terms in proposition 1.3.1 is captured in the following statement: Let \( K \) be the field of rational functions and \( T : K \to \mathbb{R}^+ \) a function which satisfies (a) - (d) without the \( O(1) \) terms and \( T_{\text{constant}} = 0 \), then \( T \) is proportional to the degree of the rational function. I.e. \( T(R) = \deg R(x) \cdot T(x) \).

## 1.5 Cartan’s Identity

The objective in this section will be to motivate the approach taken by Ahlfors and Shimizu independently to arrive at the First Main Theorem by proving a formula for \( T_f(r) \).

Consider a function \( f \) meromorphic in \( \mathbb{C} \) with \( f(0) \neq 0 \). Then Jensen’s formula, applied to \( f - e^{i\psi} \) (for fixed \( \psi \)) says

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta}) - e^{i\psi}| \, d\theta = \log |f(0) - e^{i\psi}| + N_{f - e^{i\psi}}(r, 0) - N_{f - e^{i\psi}}(r, \infty) \\
= \log |f(0) - e^{i\psi}| + N_f(r, e^{i\psi}) - N_f(r, \infty)
\]

Integrating both sides w.r.t \( \frac{d\psi}{2\pi} \) and interchanging the order of integration yields

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta}) - e^{i\psi}| \, d\theta \, d\psi = \int_{0}^{2\pi} \log |f(0) - e^{i\psi}| \, d\psi \frac{d\psi}{2\pi} + \int_{0}^{2\pi} N_f(r, e^{i\psi}) \, d\psi - N_f(r, \infty) \] (1.30)

for all values of \( \psi \) (except possibly one if \( |f(0)| = 1 \)). To simplify this, we will use the following lemma

**Lemma 1.5.1.** For all \( a \in \mathbb{C} \), we have

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log |a - e^{i\theta}| \, d\theta = \log_+ (|a|).
\] (1.31)

**Proof.** This formula immediately follows from applying Jensen’s formula to \( g(z) = z - a \), however, we present a different proof. We first consider the case \( |a| > 1 \). Note that

\[
\int_{0}^{2\pi} \log |a - e^{i\theta}| \, d\theta = \int_{0}^{2\pi} \Re \left( \log(re^{i\theta} - a) \right) \, d\theta = \Re \left( \int_{0}^{2\pi} \log(re^{i\theta} - a) \, d\theta \right)
\]
where here, the branch of the logarithm is chosen so that \( \log(\cdot) \) is holomorphic in \( \mathbb{D} \) (this is possible since \( 1 < |a| \)). Then if we write \( z = e^{i\theta} \), we arrive at

\[
\int_0^{2\pi} \log |e^{i\theta} - a| \, d\theta = \text{Re} \left( \frac{1}{i} \int_{|z|=1} \frac{\log(z-a)}{z} \, dz \right) = \text{Re} \left( 2\pi \log(-a) \right) = 2\pi \log |a|.
\]

where the integral is evaluated using the Cauchy Integral Formula. This proves the result for \( |a| > 1 \). In the case \( |a| < 1 \), note that

\[
\log |e^{i\theta} - a| = \log |a| + \log |a^{-1} - e^{-i\theta}|
\]

and so, with a change of variable the previous result holds and

\[
\int_0^{2\pi} \log |e^{i\theta} - a| = \log |a| + \int_0^{2\pi} \log |a^{-1} - e^{-i\theta}| \, d\theta = \log |a| + \log |a^{-1}| = 0
\]

Observe that if \( a \) is replaced with \( a/r \), a more general formula (which will be used later) can be proven:

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |a - re^{i\theta}| \, d\theta = \begin{cases} \log r & \text{for } |a| < r \\ \log |a| & \text{for } |a| \geq r \end{cases} \tag{1.32}
\]

Applying this lemma to (1.30) yields

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta = \log |f(0)| + \int_0^{2\pi} N_f(r, e^{i\theta}) \, d\psi - N_f(r, \infty)
\]

and so we arrive at the following theorem

**Theorem 1.5.2 (Cartan’s Identity).** Let \( f(z) \) be a function meromorphic in \( \mathbb{C} \) such that \( f(0) \neq 0 \), then

\[
T_f(r) = \log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} N_f(r, e^{i\theta}) \, d\theta \tag{1.33}
\]

**Corollary 1.5.3.** For any function meromorphic in \( \mathbb{C} \),

\[
\frac{1}{2\pi} \int_0^{2\pi} m_f(r, e^{i\theta}) \, d\theta \leq \log 2 \tag{1.34}
\]

**Proof.** From (1.7) we get that

\[
T_f(r) = N_f(r, 0) + m_f(r, 0) + \log |\eta(f)|.
\]

Applying this to \( f - e^{i\theta} \) (assuming \( f(0) \neq e^{i\theta} \)) and observing that \( m_{f-a}(r, 0) = m_f(r, a), N_{f-a}(r, 0) = N_f(r, a) \) yields

\[
T_{f-e^{i\theta}}(r) = N_f(r, e^{i\theta}) + m_f(r, e^{i\theta}) + \log |f(0) - e^{i\theta}|. \tag{1.35}
\]

Using this and

\[
|T_{f-a}(r) - T_f(r)| \leq \log |(a)| + \log 2
\]
which follows from (1.8) yields

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| N_f(r,e^{i\theta}) + m_f(r,e^{i\theta}) + \log |f(0) - e^{i\theta}| - T_f(r) \right| \, d\theta \leq \log 2
\]

Integrating and applying (1.31) yields

\[
\left| \frac{1}{2\pi} \int_0^{2\pi} N_f(r,e^{i\theta}) \, d\theta + \int_0^{2\pi} m_f(r,e^{i\theta}) \, d\theta + \log |f(0) - T_f(r)| \right| \leq \log 2
\]

where the cancellation follows from Cartan’s Identity. Finally, positivity of \( m_f(r,e^{i\theta}) \) yields the result.

**Corollary 1.5.4.** \( T_f(r) \) is convex, monotone in \( \log r \) (i.e. the map \( y \to T(e^{iy}) \) is convex and monotone).

**Proof.** Recall that

\[
N_f(r,e^{i\theta}) = n_f(0,e^{i\theta}) \log r + \sum_{|z_j| \neq 0} m_j \left( \log r - \log |z_j| \right)
\]

where \( f(z_j) = e^{i\theta}, \ z_j \in D_r \). This is a piece-wise linear, monotonically increasing function in \( \log r \) for each \( e^{i\theta} \) and so is monotone and convex in \( \log r \). By this and (1.33),

\[
T_f(e^{r_1}) + (1 - \alpha)T_f(e^{r_2}) \leq \log |f(0)| + \frac{\alpha}{2\pi} \int_0^{2\pi} N_f(e^{r_1}, e^{i\theta}) \, d\theta + \frac{1 - \alpha}{2\pi} \int_0^{2\pi} N_f(e^{r_2}, e^{i\theta}) \, d\theta
\]

1.6 Ahlfors-Shimizu Characteristic

Since integrating Jensen’s formula over circles yielded a useful expression for \( T_f(r) \), one is motivated to integrate over different sets. We will average Jensen’s formula over \( \mathbb{C} \) with respect to the normalized spherical area measure

\[
d\rho(w) = \frac{dudv}{\pi(1 + |w|^2)^2}, \quad \text{where } w = u + iv.
\]

It will be important that this measure is invariant under rotation of the sphere (see [5, Prop. 6.5.1]). Now, integrating (1.5) against \( d\rho(a) \) and interchanging the order of integration yields

\[
\int_0^{2\pi} U(f(re^{i\theta})) \, d\theta = U(\eta(f)) + \int_\mathbb{C} N_f(r,a) \, d\rho(a) - N_f(r,\infty)
\]

(1.37)
where
\[ U(w) = \int_{\mathbb{C}} \log |w - a| \, dp(a). \]

\( U(w) \) can be computed explicitly via (1.32)
\[
U(w) = \int_0^{|w|} \int_0^{2\pi} \frac{2r \log |w - re^{i\theta}|}{2\pi(1 + r^2)^2} \, drd\theta + \int_{|w|}^\infty \int_0^{2\pi} \frac{2r \log |w - re^{i\theta}|}{2\pi(1 + r^2)^2} \, drd\theta \\
= \log |w| \int_0^{|w|} \frac{2r}{1 + r^2} \, dr + \int_{|w|}^\infty \frac{2r \log r}{1 + r^2} \, dr \\
= \log |w| \cdot \frac{|w|^2}{1 + |w|^2} + \log |w| \cdot \frac{1}{1 + |w|^2} - \log \left( \frac{|w|}{\sqrt{1 + |w|^2}} \right) = \log \sqrt{1 + |w|^2}
\]

\[ \implies U(w) = \log \sigma(w, \infty)^{-1} \]

where the final integral was computed by parts and
\[
\sigma(z, w) = \begin{cases} 
\frac{|w - z|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}, & w, x \in \mathbb{C} \\
\frac{1}{\sqrt{1 + |z|^2}}, & z \in \mathbb{C}, w = \infty \\
0, & z = w = \infty
\end{cases} \quad (1.38)
\]
is the chordal distance on the sphere (also rotation invariant). Applying this yields
\[
\frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\sigma(f(re^{i\theta}), \infty)} \, d\theta = \log \frac{1}{\sigma(\eta(f), \infty)} + \int_{\mathbb{C}} N_f(r, a) \, dp(a) - N_f(r, \infty) \quad (1.39)
\]

**Definition 1.6.1.** The Ahlfors-Shimizu proximity function is defined to be
\[ \hat{m}_f(r, a) := \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{\sigma(f(re^{i\theta}), a)} \, d\theta \quad (1.40) \]

**Definition 1.6.2.** The Ahlfors-Shimizu characteristic is defined to be
\[ \hat{T}_f(r, a) := \int_{\mathbb{C}} N_f(r, a) \, dp(a) \quad (1.41) \]

With this notation, (1.39) becomes
\[ N_f(r, \infty) + \hat{m}_f(r, \infty) = \log \frac{1}{\sigma(\eta(f), \infty)} + \hat{T}_f(r). \quad (1.42) \]

Then, if \( f(0) \neq 0, \infty, \) (1.39) can be written as
\[ \hat{m}_f(r, \infty) + N_f(r, \infty) = \hat{m}_f(0, \infty) + \hat{T}_f(r). \quad (1.43) \]

Furthermore, observe that
\[ \left| \log \sqrt{1 + x^2} - \log_+ |x| \right| \leq \frac{1}{2} \log 2 \]
1.6. AHLFORS-SHIMIZU CHARACTERISTIC

since for \(0 < |x| \leq 1\), \(\log_+ |x| = 0\) and the result is immediate, while for \(|x| > 1\),

\[
\log_+ \left( \frac{\sqrt{1 + x^2}}{|x|} \right) = \log \sqrt{1 + \frac{1}{x^2}} \leq \frac{1}{2} \log 2.
\]

Integration immediately yields

\[
|\hat{m}_f(r, \infty) - m_f(r, \infty)| \leq \frac{1}{2} \log 2.
\] (1.44)

Hence, we have shown that

\[
\hat{T}(r) = T(r) + O(1).
\] (1.45)

The following theorem can be used to establish a different formula for \(\hat{T}(r)\).

**Theorem 1.6.3.** Let \(d\rho(a) = G(a) \, du \, dv\), \((a = u + iv)\) where \(G(a)\) is strictly positive and continuous for \(a \in \mathbb{C} \setminus \{a_j\}_{j=1}^n\), \(n < \infty\). Then, for any function \(f\) meromorphic in \(\mathbb{C}\),

\[
\int_{\mathbb{C}} n_f(r, a) \, d\rho(a) = \int_{|z| < r} |f'(z)|^2 G(f(z)) \, dxdy, \quad z = x + iy.
\] (1.46)

**Proof.** Note that if \(f\) were constant, then \(n_f(r, a) = 0\) for all but one value of \(a\) (a set of measure 0), and hence the LHS is identically 0, but so is the RHS since \(f' \equiv 0\). Hence, suppose \(f(z)\) is not constant. Then there exists a finite sequence \(\{z_j\}_{j=1}^k \subset \mathcal{D}_r(0)\) such that \(f(z) < \infty\), \(f'(z) \neq 0\) on \(\mathcal{D}_r(0) \setminus \{z_j\}_{j=1}^k\). Since \(\mathcal{D}_r(0) \setminus \{z_j\}_{j=1}^k\) is compact, it can be covered by countably many open disks \(\{D_j\}_{j=1}^n\) such that \(f\) is a bijection on \(D_j\) (\(f'(z_0) \neq 0 \implies\) locally invertible). Let

\[
U_j := \left( D_j \setminus \bigcup_{k=1}^{j-1} D_k \right) \cap \mathcal{D}_r(0)
\]

so that

\[
\sum_{j=1}^\infty \int_{U_j} |f'(z)|^2 G(f(z)) \, dxdy = \sum_{j=1}^\infty \int_{f(U_j)} G(a) \, dA
\]

since \(f(z)\) is a bijection and the Jacobian of this transformation is \(|f'(z)|^2\). Since Each \(a\) appears in \(f(U_j)\) (up to a set of measure zero due to boundaries of \(D_j\)’s being excluded) \(n_f(r, a)\) times. Hence,

\[
\sum_{j=1}^\infty \int_{f(U_j)} G(a) \, dA = \int_{\mathbb{C}} n_f(r, a) \, d\rho(a).
\] (1.47)

Theorem 1.6.3 allows us to write

\[
\hat{T}(r) = \int_0^r \frac{A(t)}{t} \, dt, \quad \text{where} \quad A(t) = \frac{1}{\pi} \int_{|z| \leq t} \frac{|f'|^2}{(1 + |f|^2)^2} \, dA.
\] (1.48)

The function \(A(t)\) is to be understood as the average covering number. I.e. the average number of times a point in \(\hat{\mathbb{C}}\) is covered by \(f : \mathcal{D}_t(0) \rightarrow \hat{\mathbb{C}}\).

In this setting, it is much easier to prove a First Main Theorem-like statement.
Theorem 1.6.4 (Ahlfors-Shimizu FMT). It holds that
\[ N_f(r, a) + \hat{m}_f(r, a) = \hat{T}_f(r) + \hat{m}_f(0, a) \] (1.49)
\[ = \hat{T}_f(r) + O(1) \] (1.50)
\[ = T_f(r) + O(1) \] (1.51)

Proof. Fix \( a \in \hat{\mathbb{C}} \), \( a \neq \infty \) and let \( R \) be a linear fractional transformation such that \( R(a) = \infty \). Since \( \hat{T}_f(r) \) is rotation invariant, we have on the one hand that
\[ \hat{T}_Rf(r) = \hat{T}_f(r). \] (1.52)
On the other hand, since chordal distance is rotation-invariant,
\[ \hat{m}_f(r, a) = \hat{m}_{RF}(r, \infty), \quad N_f(r, a) = N_{RF}(r, \infty) \] (1.53)
which yields
\[ N_f(r, a) + \hat{m}_f(r, a) = \hat{T}_{RF}(r) = \hat{T}_f(r). \]
The rest follows from (1.45) and independence of \( \hat{m}_f(0, a) \) of \( r \). \( \square \)
Chapter 2

The Second Main Theorem

To state the second main theorem, we need some definitions.

Definition 2.0.1. If $f$ is a function meromorphic in $\mathbb{C}$, the point $z_0$ is a critical point if either

(a) $f(z_0) < \infty$ and $f(z) - f(z_0)$ has a higher-order zero at $z_0$, namely

$$f(z) - f(z_0) \sim C(z - z_0)^{l+1}$$

(b) $f(z_0) = \infty$ and $f$ has a pole of order $l + 1 \geq 2$.

In both cases, $l$ is the order of the critical point. $n_{1,f}(r)$ is the number of critical points, counting multiplicity, in $D_r(0)$. $N_{1,f}(r)$ is defined similarly to $N_f(r,a)$ as

$$N_{1,f}(r) = \int_0^r \frac{n_{1,f}(s) - n_{1,f}(0)}{s} \mathrm{d}s + n_{1,f}(0) \log r. \quad (2.1)$$

 Proposition 2.0.2. It holds that

$$N_{1,f}(r) = N_{f'}(r,0) + 2N_f(r,\infty) - N_{f'}(r,\infty) \quad (2.2)$$

Proof. If $z_0$ is of type (a), then $f'(z_0) = 0$, a zero of order $l$. This is counted by $N_{f'}(r,0)$ only.

If $f(z_0) = \infty$ and $f(z_0) = C(z - z_0)^{-(l+1)}$, then $2N_f(r,\infty)$ contributes $2(l + 1) \log_+(r/|z_0|)$, while $N_{f'}(r,\infty)$ contributes $(l + 2) \log_+(r/|z_0|)$. Hence, total contribution is

$$[2(l + 1) - (l + 2)] \log_+(r/|z_0|) = l \log_+(r/|z_0|).$$

\[ \Box \]

Theorem 2.0.3 (Second Main Theorem). For any $\{a_j\}_{j=1}^q$ distinct points in $\hat{\mathbb{C}}$, there is a subset $E \subset (0,\infty)$ of finite measure so that as $r \to \infty$, $r \not\in E$,

$$\sum_{j=1}^q m_f(r,a_j) + N_{1,f}(r) \leq 2T_f(r) + S(r) \quad (2.3)$$

where $S(r) = O((\log(rT_f(r)))$.

Nevanlinna and Ahlfors both had their own, different proofs of the SMT, and we will explore both in the next two sections, beginning with Nevanlinna’s approach.
2.1 SMT: Nevanlinna’s Proof

We begin by the following calculation. Suppose $|a_j| < \infty \forall j = 1, \ldots, q$ and consider the function
\[
g(z) := \sum_{j=1}^{q} \frac{1}{f(z) - a_j}. \tag{2.4}
\]

On the one hand, by $1.4.1$ and the FMT
\[
m_g(r, \infty) = T_g(r) - N_g(r, \infty) = qT_f(r) - \sum_{j=1}^{q} N_f(r, a_j) + O(1) = \sum_{j=1}^{q} m_f(r, a_j) + O(1). \tag{2.5}
\]

On the other hand, by lemma $1.2.2$
\[
m_g(r, \infty) = m_g f'/f' \leq m_1 f'(r, \infty) + \sum_{j=1}^{q} m_{f'/f-a_j}(r, \infty) + \ln q \tag{2.6}
\]

Putting (2.5) and (2.7) together yields
\[
\sum_{j=1}^{q} m_f(r, a_j) \leq m_1 f'(r, \infty) + \sum_{j=1}^{q} m_{f'/f-a_j}(r, \infty) + \ln q \tag{2.8}
\]

This in turn implies
\[
m(r, \infty) + \sum_{j=1}^{q} m(r, a_j) + N_1, f(r) \leq m_f(r, \infty) + N_1,f(r) + m_1 f'(r, \infty) + \sum_{j=1}^{q} m_{f'/f-a_j}(r, \infty) + O(1) \tag{2.9}
\]

and by application of the FMT and proposition $2.0.2$ we get
\[
m(r, \infty) + \sum_{j=1}^{q} m(r, a_j) + N_1, f(r) \leq T_f(r) + N_f(r, \infty) + m(r, f') + \sum_{j=1}^{q} m_{f'/f-a_j}(r, \infty) + O(1) \tag{2.10}
\]

\[
\leq 2T_f(r) + m_{f'/f}(r, \infty) + \sum_{j=1}^{q} m_{f'/f-a_j}(r, \infty) + O(1) \tag{2.11}
\]

It now remains to estimate the last two terms and the proof is done. Note that this statement is equivalent to theorem $2.1.5$ since one can consider $q + 1$ distinct $a_j$’s, where $a_{q+1} = \infty$ and the rest, being distinct from $a_{q+1}$, must then be finite. To do so, we prove some preliminary results

**Theorem 2.1.1.** Let $f(z)$ be a meromorphic function satisfying $f(0) = 1$. For all $r$ and $R$ satisfying $1 < r < R < \infty$ and all $0 < \alpha < 1$, we have

\[
m_{f'/f}(r, \infty) \leq \frac{1}{\alpha} \log^+ T_f(R) + \max \left\{ 2, \frac{1}{\alpha} \right\} \left( \log \frac{1}{R-r} + \log R \right) + \frac{1}{\alpha} \log \frac{64}{1-\alpha} \tag{2.12}
\]

To that end, we first prove a lemma
Lemma 2.1.2. Let $f(x), g(x)$ be non-negative measurable functions on $[a, b]$ and let

$$A = \int_a^b g(x) \, dx > 0$$

then,

$$\frac{1}{A} \int_a^b (\log_+ f) \cdot g \, dx \leq \log_+ \left( \frac{1}{A} \int_a^b f(x)g(x) \, dx \right) + \log 2. \quad (2.14)$$

Proof. Set

$$m := \frac{1}{A} \int_a^b \max \{ f(x), 1 \} g(x) \, dx, \ \phi(x) := \max \{ f(x), 1 \} - m,$$

Then it is immediate that $m \geq 1$ and

$$\int_a^b \phi \cdot g \, dx = Am - Am = 0.$$

Next,

$$\frac{1}{A} \int_a^b (\log_+ f) \cdot g \, dx = \frac{1}{A} \int_a^b \log \left( \max \{ f(x), 1 \} \right) \cdot g \, dx = \frac{1}{A} \int_a^b \log(\phi + m) \cdot g \, dx = \log m + \frac{1}{A} \int_a^b \frac{\phi}{m} \cdot g \, dx = \log_+ m (m \geq 1)$$

This implies

$$\frac{1}{A} \int_a^b (\log_+ f) \cdot g \, dx \leq \log_+ \left( \frac{1}{A} \int_a^b \max \{ f(x), 1 \} g(x) \, dx \right)$$

$$\leq \log_+ \left( \frac{1}{A} \int_a^b (1 + f(x)) g(x) \, dx \right) = \log_+ \left( \frac{1}{A} \int_a^b f \cdot g \, dx + 1 \right) \leq \log_+ \left( \frac{1}{A} \int_a^b f \cdot g \, dx \right) + \log 2$$

where the last inequality follows from lemma 1.2.2(a). \hfill \Box

We are now ready to prove our theorem

Proof (of theorem 2.1.1). To begin with, recall that

$$\ln f(z) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(se^{i\theta})| \frac{se^{i\theta} + z}{se^{i\theta} - z} \, d\theta - \sum_{|z_j| < s} \ln \frac{s^2 - z_j^2}{s(z - z_j)}$$

$$+ \sum_{|w_j| < s} \ln \frac{s^2 - w_j^2}{s(z - w_j)} \quad (2.15)$$

This is the complex Poisson-Jensen formula (see [5, Theorem 9.8.2]). Differentiating both sides w.r.t. $z$ yields, for $|z| < s$

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(se^{i\theta})| \frac{2 \cdot se^{i\theta}}{(se^{i\theta} - z)^2} \, d\theta + \sum_{|z_j| < s} \left( \frac{s^2 - z_j^2}{(s^2 - z_j^2)(z - z_j)} \right)$$

$$- \sum_{|w_j| < s} \left( \frac{s^2 - w_j^2}{(s^2 - w_j^2)(z - w_j)} \right) \quad (2.16)$$
Next, observe that for $|z_j| < s$ (respectively, $|w_j| < s$)

$$\left| \frac{s^2 - |z_j|^2}{(s^2 - |z_j|^2)(z - z_j)} \right| \leq \frac{s^2 - |z_j|^2}{(s^2 - |z_j|^2)(|z - z_j|)} = \frac{s + |z_j|}{s} \cdot \frac{1}{|z - z_j|} \leq \frac{2}{|z - z_j|}.$$  

with similar inequality when $z_j$'s are replaced with $w_j$. Furthermore,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(se^{i\theta})| \frac{2 \cdot se^{i\theta}}{(se^{i\theta} - z)^2} \, d\theta \leq \frac{2s}{(s - |z|)^2} \frac{1}{2\pi} \int_0^{2\pi} \log |f(se^{i\theta})| \, d\theta$$

$$= \frac{2s}{(s - |z|)^2} (m_f(s, \infty) + m_{1/f}(s, \infty))$$

$$\leq \frac{4s}{(s - |z|)^2} T_f(s)$$

where the last inequality follows from the FMT. Hence,

$$\left| \frac{f'(z)}{f(z)} \right| \leq \frac{4s}{(s - |z|)^2} T_f(s) + 2 \sum_{|z_j| < s} \frac{1}{|z - z_j|} + 2 \sum_{|w_j| < s} \frac{1}{|z - w_j|} \quad (2.17)$$

Putting this together yields

$$\int_0^{2\pi} \left| \frac{f(re^{i\varphi})}{f(re^{i\varphi})} \right|^\alpha \, d\varphi \leq \int_0^{2\pi} \left| \frac{4s}{(s - r)^2} T_f(s) + 2 \sum_{|z_j| < s} \frac{1}{|re^{i\varphi} - z_j|} + 2 \sum_{|w_j| < s} \frac{1}{|re^{i\varphi} - w_j|} \right|^\alpha \, d\varphi$$

$$\leq \int_0^{2\pi} \left( \frac{4^\alpha s^\alpha}{(s - r)^{2\alpha}} T_f^\alpha(s) + 2^\alpha \sum_{|z_j| < s} \frac{1}{|re^{i\varphi} - z_j|^\alpha} + 2^\alpha \sum_{|w_j| < s} \frac{1}{|re^{i\varphi} - w_j|^\alpha} \right) \, d\varphi$$

$$= 2\pi \cdot \frac{4^\alpha s^\alpha T_f^\alpha(s)}{(s - r)^{2\alpha}} + 2^\alpha \int_0^{2\pi} \sum_{|z_j| < s} \frac{1}{|re^{i\varphi} - z_j|^\alpha} + \sum_{|w_j| < s} \frac{1}{|re^{i\varphi} - w_j|^\alpha} \, d\varphi$$

We next estimate the remaining integrals (we estimate integrals with $z_j$’s with with the understanding that the same estimates for $w_j$ integrals)

$$\int_0^{2\pi} \frac{d\varphi}{|re^{i\varphi} - z_j|^\alpha} \leq \int_0^{2\pi} \frac{d\varphi}{|re^{i\varphi} - |z_j||^\alpha}$$

$$\leq \int_0^{2\pi} \frac{d\theta}{r \sin \varphi} \leq \frac{4}{r^\alpha} \cdot \frac{\pi}{2} \cdot \frac{\pi^{\alpha/2} \, d\varphi}{\varphi^{\alpha}}$$

which gives us

$$\int_0^{2\pi} \frac{d\varphi}{|re^{i\varphi} - z_j|^\alpha} \leq \frac{2\pi}{r^\alpha (1 - \alpha)}$$

and hence,

$$\int_0^{2\pi} \left| \frac{f(re^{i\varphi})}{f(re^{i\varphi})} \right|^\alpha \, d\varphi \leq 2\pi \cdot \frac{4^\alpha s^\alpha T_f^\alpha(s)}{(s - r)^{2\alpha}} + (n_f(s, 0) + n_{1/f}(s, \infty)) \frac{2^\alpha \cdot 2\pi}{r^\alpha (1 - \alpha)} \quad (2.18)$$
2.1. SMT: NEVANLINNA’S PROOF

Since $1 < r < R$, we have that

$$n_f(r, a) \leq \frac{N_f(R, a)}{\log(R/r)} \leq \frac{R}{R - r} N_f(R, a). \quad (2.19)$$

Indeed, this follows from the definition of $n_f(r, a)$ since

$$N_f(R, a) \geq \int_r^R \frac{n_f(s, a) - n_f(0, a)}{s} ds + n_f(0, a) \log R$$

$$\geq (n_f(r, a) - n_f(0, a)) \log R/r + n_f(0, a) \log R = n_f(r, a) \log R/r + n_f(0, a) \log r$$

$$\geq n_f(r, a) \log R/r = n_f(r, a) \int_r^R \frac{ds}{s} \geq \frac{R}{R - r} \cdot n_f(r, a).$$

Letting $s = (r + R)/2$ ($r < s < R$) and applying (2.19) yields

$$\int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^\alpha \, d\theta \leq 4^\alpha s^\alpha T_f^\alpha(s) + \frac{2R}{R - r} \cdot \frac{2^\alpha - 2}{r^\alpha(1 - \alpha)} (N_f(R, 0) + N_f(R, \infty))$$

$$\leq 4^\alpha s^\alpha T_f^\alpha(s) + \frac{4R}{R - r} \cdot \frac{2^\alpha - 2}{r^\alpha(1 - \alpha)} T_f(R)$$

$$\leq \frac{16}{1 - \alpha} \max\{T_f(R), 1\} \left( \frac{1}{R^\alpha} \left(1 - \frac{r}{R}\right)^{2\alpha} + \frac{1}{r^\alpha} \left(1 - \frac{r}{R}\right)^{-1}\right)$$

$$\leq \frac{32}{1 - \alpha} \max\{T_f(R), 1\} \left(1 - \frac{r}{R}\right)^{-\max\{2\alpha, 1\}}$$

By the previous lemma,

$$m_{f/f}(r, \infty) = \frac{1}{\alpha} \cdot 2\pi \int_0^{2\pi} \log_+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^\alpha \, d\theta \leq \frac{1}{\alpha} \left( \log_+ \left[ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^\alpha \, d\theta \right] + \log 2 \right)$$

which gives

$$m_{f/f}(r, \infty) \leq \frac{1}{\alpha} \left[ \log_+ \left( \frac{32}{1 - \alpha} \max\{T_f(R), 1\} \left(1 - \frac{r}{R}\right)^{-\max\{2\alpha, 1\}} \right) + \log 2 \right] \quad (2.20)$$

Finally, applying lemma 1.2.2 (c) yields the result.

Now, we would like to pass to an upper bound of $m_{f/f}(r, \infty)$ in terms of $T_f(r)$ instead of $T_f(R)$, $r < R$. The following theorem is the tool we need

**Theorem 2.1.3** (Borel-Nevanlinna). Let $u(r)$ be a continuous, non-decreasing function on $[r_0, \infty)$, tending to $+\infty$ as $r \to \infty$. Let $\varphi(u)$ be a continuous positive non-decreasing function on $[u_0, \infty)$, $u_0 = u(r_0)$, having zero limit as $u \to \infty$ and satisfying

$$\int_{u_0}^\infty \varphi(u) \, du < \infty. \quad (2.21)$$

Then for all $r \geq r_0$ except possibly, a set of finite measure the inequality

$$u(r + \varphi(u(r))) < u(r) + 1 \quad (2.22)$$

**Proof.** Let $E \subset [r_0, \infty)$ be the closed set where (2.22) does not hold and let $E \cap [r, \infty) = E(r)$. We assume that for all $r \geq r_0$, $E(r) \neq \emptyset$, for otherwise the result is trivial.
Let \( r_1 = \min_{r \in E} r \). Let \( r'_1 \) be the least value of \( r \) satisfying \( u(r) = u(r_1) + 1 \). Since \( u(r) \) is non-decreasing, \( r'_1 > r_1 \). On the other hand, by the definition of \( E \),
\[
    u \left( r_1 + \varphi(u(r_1)) \right) \geq u(r_1) + 1 = u(r'_1).
\]

Now, since \( r'_1 > r_1 \) and \( u(r) \) is non-decreasing, we have that \( r_1 + \varphi(u(r_1)) \geq r'_1 \Leftrightarrow r'_1 - r_1 \leq \varphi(u(r_1)) \).

Suppose we have generated the values \( r_1, r'_1, r_1', ..., r_n' \). Set \( r_{n+1} = \min_{r \in E(r'_n)} r \) and let \( r'_{n+1} \) be the least number \( r \) satisfying \( u(r) = u(r_{n+1}) + 1 \) and \( r'_{n+1} > r_{n+1} \).

We have
\[
    u(r_{n+1} + \varphi(u(r_{n+1}))) \geq u(r_{n+1}) + 1 = u(r'_{n+1}) \Leftrightarrow r'_{n+1} - r_{n+1} \leq \varphi(u(r_{n+1})).
\]

Since \( u(r_{n+1}) \geq u(r'_n) = u(r_n) + 1 \), we get that \( u(r_{n+1}) - u(r_n) \geq 1 \Leftrightarrow u(r_{n+1}) - u(r_1) \geq n \Rightarrow u(r_{n+1}) \geq u_0 + n \). Hence, we have that \( u(r_n) \to \infty \) and since \( u \) is continuous, non-decreasing, \( r_n \to \infty \) as \( n \to \infty \).

By the definition of \( \{r_n\} \), we have that (2.22) holds on \((r'_{n-1}, r_n), \ n = 1, 2, ...\) where \( r_0 = r'_0 \). Thus, \( E \) is covered by \([r_n, r'_n]\) and
\[
    \sum_{n=1}^{\infty} (r'_n - r_n) \leq \sum_{n=1}^{\infty} \varphi(u(r_n)) \leq \sum_{n=1}^{\infty} \varphi(u_0 + n - 1) \leq \varphi(u_0) + \int_{u_0}^{\infty} \varphi(u) \ du < \infty.
\]

Note that we have proved that \( E \) is covered by pair-wise disjoint intervals with finite total length, whose endpoints escape to infinity.

R. Nevanlinna showed that condition (2.21) cannot be removed for suppose \( \varphi(u) \) were to satisfy all the other conditions of the above theorem and
\[
    \int_{u_0}^{\infty} \varphi(u) \ du = \infty. \quad (2.23)
\]

Set
\[
    r(u) = \int_{u_0}^{u} \varphi(t) \ dt, \ u \geq u_0.
\]

Then \( u(r) \), defined as the inverse of \( r(u) \), is defined on \([0, \infty)\) and will serve as the counter-example. By mean value theorem,
\[
    u \left( r + \varphi(u(r)) \right) = u(r_0) \left( r + \theta \varphi(u(r)) \right) \varphi(u(r)),
\]
where \( 0 < \theta = \theta(r) < 1 \). On the other hand,
\[
    u'(r + \theta \varphi(u(r))) = \frac{1}{u'(r + \theta \varphi(u(r)))} = \frac{1}{\varphi(u(r) + \theta \varphi(u(r)))} \geq \frac{1}{\varphi(u(r))}
\]
and hence,
\[
    u(r + \varphi(u(r))) - u(r) \geq 1
\]
which is the opposite inequality we had hoped for (in fact, the inequality we desired holds nowhere on \([0, \infty)!)\).

**Theorem 2.1.4** (Lemma on the logarithmic derivative). Each meromorphic function \( f(z) \) satisfies
\[
    m_{f'/f}(r, \infty) = O(\log(rT_f(r))) \quad (2.24)
\]
except, possibly, for a set \( E \) of finite total length.
2.1. SMT: NEVANLINNA’S PROOF

Proof. Observe that WLOG, we can assume that \( f(0) = 1 \). Indeed, in the general case, write \( f(z) = Az^p \varphi(z) \) where \( A \in \mathbb{C}, \ p \in \mathbb{Z} \) and \( \varphi(0) = 1 \). If we assume the theorem for \( \varphi(z) \), we get that

\[
m \left( \frac{f'}{f} ; r, \infty \right) = m \left( \frac{p}{z} + \frac{\varphi'}{\varphi} ; r, \infty \right) = m \left( \frac{\varphi'}{\varphi} ; r, \infty \right) + O(1) = O(\log(rT_f(r))).
\]

Hence, assume \( f(0) = 1 \) and that \( T_f(r) > 0 \) for \( r \geq r_0 > 1 \). Letting \( R > r, \ \alpha = 1/2 \) in (2.13), we get that for \( r \geq r_0 \)

\[
m_{f'/f}(r, \infty) \leq 2\log T(R, f) + 2\log \left( \frac{1}{R - r} \right) + 2\log R + 2\log 128 \tag{2.25}
\]

Set \( R = r + \frac{1}{T_f'}(r) \leq 2r \). By theorem 2.1.3 we get that, except on a set of finite measure,

\[
T_f(R) = T(r + T_f'^{-2}(r)) < T_f(r) + 1 \tag{2.26}
\]

(apply the theorem with \( u(r) = T_f(r) \) and \( \varphi(u) = u^{-2} \)). (2.25) becomes

\[
m_{f'/f}(r, \infty) \leq 6 \log T_f(r) + 2\log r + O(1). \tag{2.27}
\]

Note that in the case where \( f(z) \) is rational, we know that \( T_f(r) = O(\log r) \) and so, plugging \( R = 2r \) into (2.25) immediately spits out

\[
m_{f'/f}(r) = O(\log r)
\]

for all \( r \). It is also immediate that \( \forall k > 1, \)

\[
m_{f'/f}(r, \infty) = O(\log(rT_f(kr))), \ \ r \to \infty
\]

without the need of an exceptional set. This follows by letting \( R = kr \) in (2.25). The question is, can the exceptional set be dropped. The answer is negative.

Returning to our main objective and applying the previous result to (2.12) yields the following theorem

**Theorem 2.1.5** (Second Main Theorem). Let \( f(z) \) be a non-identically constant function meromorphic in \( \mathbb{C} \) and let \( a_1, \ldots, a_q \in \hat{\mathbb{C}} \) be distinct. Then

\[
\sum_{n=1}^{q} m_f(r, a_n) + N_{1,f}(r) \leq 2T_f(r) + S_f(r) \tag{2.28}
\]

where \( S_f(r) = O(\log(rT_f(r))) \).

Furthermore, from the lemma on logarithmic derivatives, one can deduce that

\[
T_{f'/f}(r) = N_{f'}(r, \infty) + m_{f'}(r, \infty)
\]

\[
\leq 2N_f(r, \infty) + m_{f'/f}(r, \infty) + m_f(r, \infty)
\]

\[
\leq 2N_f(r, \infty) + m_f(r, \infty) + O(\log(rT_f(r)))
\]

\[
\leq 2T_f(r) + O(\log(rT_f(r))).
\]

From here, it follows by induction and the observation that \( N_{f^{(n)}}(r, \infty) \leq (n + 1)N_f(r, \infty) \) that

\[
T_{f^{(l)}}(r) \leq (l + 1)T_f(r) + O(\log(rT_f(r))) \tag{2.29}
\]
2.2 SMT: Ahlfors’s Proof

Ahlfors’s reused the idea of averaging Jensen’s formula,

\[ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta}) - a| \, d\theta = \log |f(0) - a| + N_f(r, a) - N_f(r, \infty) \]  

(2.30)

but with respect to the measure

\[ dp(a) = \frac{p^2(a)}{(1 + |a|^2)^2} \, da, \quad p(a) = C \left( \prod_{j=1}^q \sigma(a, a_j) \sum_{j=1}^q \log(\sigma(a, a_j)) \right)^{-1} \]  

(2.31)

where \( \sigma \) is as in (1.38) and \( C \) is a constant ensuring \( dp(a) \) is a probability measure. Furthermore, observe that from lemma 1.2.2 (b)

\[ \log |f(re^{i\theta})| - 1| \leq \log_+ |f(re^{i\theta})| + \log_+ |a| \]

which, after integration, yields

\[ O(1) + m_f(r, \infty) \geq \int_C N_f(r, a) \, dp(a) - N_f(r, \infty) + O(1) \]  

(2.32)

\[ \implies \int_C N_f(r, a) \leq T_f(r) + O(1) \]  

(2.33)

Since \( n_f(0, a) = 0 \) for all but one value of \( a \), and by (1.48), we can write

\[ \int_C N_f(r, a) \, dp(a) = \int_0^r \frac{1}{7} \int_0^a \frac{p^2 \left( f(se^{i\theta}) \right)}{\left( 1 + |f(se^{i\theta})|^2 \right)^2} \left| f'(se^{i\theta}) \right|^2 \cdot \frac{8}{2\pi} \, d\theta \, ds \, dt \]  

(2.34)

\[ = \int_0^r \frac{1}{7} \int_0^a s \lambda(s) \, ds \, dt, \]  

(2.35)

where

\[ \lambda(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p^2 \left( f(re^{i\theta}) \right)}{\left( 1 + |f(re^{i\theta})|^2 \right)^2} \cdot \left| f'(re^{i\theta}) \right|^2 \, d\theta \]

By Jensen’s formula (see [6, Theorem 5.3.14]),

\[ \log \lambda(r) \geq \frac{1}{2\pi} \int_0^{2\pi} \log \left( \frac{p^2 \left( f(re^{i\theta}) \right)}{\left( 1 + |f(re^{i\theta})|^2 \right)^2} \cdot \left| f'(re^{i\theta}) \right|^2 \right) \, d\theta \]

in which we split the right hand side into 3 terms. Set \( w = f(re^{i\theta}) \) and

\[ h_1(r) := \frac{1}{\pi} \int_0^{2\pi} \log(p(w)) \, d\theta, \quad h_2(r) := -\frac{1}{\pi} \int_0^{2\pi} \log(1 + |w|^2) \, d\theta, \quad \lambda_3(r) := \frac{1}{\pi} \int_0^{2\pi} \log |w'| \, d\theta \]  

(2.36)

Then we have

\[ \log \lambda(r) \geq h_1(r) + h_2(r) + h_3(r) \]  

(2.37)

and we set out to estimate these three integrals and we begin with \( h_1(r) \). Recall that

\[ \tilde{m}_f(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log \left( \sigma(w, a)^{-1} \right) \, d\theta \]

This combined with

\[ \log p(w) = \sum_{j=1}^q \log \left( \sigma(w, a_j)^{-1} \right) - 2\log \left( \sum_{j=1}^q \log \left( \sigma(w, a_j)^{-1} \right) \right) + \log C \]  

(2.38)
2.2. SMT: AHLFOR’S PROOF

yields

\[ h_1(r) = 2 \sum_{i=1}^{q} \bar{m}_j(r, a_i) - \frac{4}{2\pi} \int_{0}^{2\pi} \log \left( \sum_{j=1}^{q} \log \left( \sigma(w, a_j)^{-1} \right) \right) d\theta + O(1). \]  

Finally,

\[ -\frac{4}{2\pi} \int_{0}^{2\pi} \log \left( \sum_{j=1}^{q} \log \left( \sigma(w, a_j)^{-1} \right) \right) d\theta \geq -4 \log \frac{1}{2\pi} \int_{0}^{2\pi} \log \left( \sum_{j=1}^{q} \log \left( \sigma(w, a_j)^{-1} \right) \right) d\theta \]

\[ \geq -4 \log \frac{1}{\pi} \int_{0}^{2\pi} \log \left( \sum_{j=1}^{q} \log \left( \sigma(w, a_j)^{-1} \right) \right) d\theta \]

\[ \geq O(\log T_f(r)) \]  

where (2.41) is by Jensen’s inequality and (2.42) is by \( \bar{m}_f(r) \leq \bar{T}_f(r) = T_f(r) + O(1) \).

We now have

\[ h_1(r) \geq 2 \sum_{i=1}^{q} \bar{m}_j(r, a_j) + O(\log T_f(r)). \]  

Next is \( h_2(r) \). Observe that since \( \sigma(w, \infty) = \sqrt{1 + w^2} \), we have that

\[ h_2(r) = -4\bar{m}_f(r, \infty). \]  

As for \( h_3(r) \), observe that by Jensen’s formula (applies to \( f'(z) \)) we have

\[ h_3(r) = 2N_f(r, 0) - N_f(r, \infty) + O(1). \]  

putting (2.43), (2.44), and (2.45) yields

\[ \log \lambda(s) \geq 2 \sum_{i=1}^{q} \bar{m}_j(s, a_i) + 2N_f(s, 0) - 2N_f(s, \infty) - 4\bar{m}_f(s, \infty) + O(\log T_f(s)). \]  

To proceed, we need the following lemma

**Lemma 2.2.1.** For \( g(x) : [c, \infty) \to \mathbb{R} \) such that \( g(x) \) is increasing and \( g(c) > 0 \), then

\[ g'(x) \leq (g(x))^{1+\epsilon} \]

holds for \( x \notin E, |E| < \infty \).

**Proof.** Let \( E \) be the set such that \( g'(x) > (g(x))^{1+\epsilon} \). Then

\[ |E| = \int_{E} dx < \int_{E} \frac{g'(x)}{g^{1+\epsilon}(x)} dx \leq \int_{c}^{\infty} \frac{g'(x)}{g^{1+\epsilon}(x)} dx = \int_{g(c)}^{\infty} \frac{dy}{y^{1+\epsilon}} = \frac{g^{-\epsilon}(c)}{\epsilon} < \infty. \]

Applying this lemma yields

\[ t\lambda(t) \leq \left( \int_{0}^{t} s\lambda(s) ds \right)^{1+\epsilon}, t \notin E, t \in [t_0, \infty) \]  

(2.47)
for some $t_0 > 0$. Applying the lemma again yields
\[
\left( \frac{r\lambda(r)}{r} \right)^{\frac{1}{1+\epsilon}} \leq \int_0^{t_0} \frac{1}{t} \left( t\lambda(t) \right)^{\frac{1}{1+\epsilon}} \, dt \leq \int_0^{t_0} \frac{1}{t} \int_t^{t_0} s\lambda(s) \, ds
\] (2.48)
and by equations (2.33), (2.35), we have
\[
\left( \frac{r\lambda(r)}{r} \right)^{\frac{1}{1+\epsilon}} \leq T_f(r) + O(1)
\] (2.49)
Plugging this into (2.46) and using Proposition 2.0.2 and the fact that $\tilde{m}_f(r, \infty) = T_f(r) - N_f(r, \infty) + O(1)$ yields
\[
\sum_{i=1}^{q} \tilde{m}(r, a_i) + N_{1,1}(r) \leq 2T_f(r) + O(\log(rT_f(r)))
\] (2.50)
Finally, noting that by the FMT, combined with $\tilde{T}_f(r) = T_f(r) + O(1)$, we have $|\tilde{m}_f(r, a) - m_f(r, a)| \leq O(1)$. This finally yields the desired estimate
\[
\sum_{i=1}^{q} m(r, a_i) + N_{1,1}(r) \leq 2T_f(r) + O(\log(rT_f(r)))
\] (2.51)

2.3 The Lemma on Logarithmic Derivatives Revisited

It turns out that the Lemma on Logarithmic Derivatives is a versatile tool on its own. To illustrate this, we consider applications to differential equations. To this end, let
\[
R(x, z) = \frac{a_0(x) + a_1(x)z + \cdots + a_n(x)z^n}{b_0(x) + b_1(x)z + \cdots + b_m(x)z^m},
\] (2.52)
where the degree of $R$ is denoted $d := \max\{m, n\}$. Furthermore, let $R_{1}(x, z)$ be rational function defined similarly to $R$ with degree $d_1$ with coefficients $\tilde{a}_i(x), \tilde{b}_i(x)$.

**Theorem 2.3.1** ([8], [9]). If
\[
R \left( x, \frac{d^n y}{dx^n} \right) = R_{1}(x, y)
\] (2.53)
admits at least one transcendental (non-rational) meromorphic solution $y(x)$, and
\[
T_{a_1}(r), T_{b_1}(r), T_{a_2}(r), T_{b_2}(r) = o(T_{y(x)}(r))
\] then
\[
(n + 1)d \geq d_1
\] (2.54)

**Remark 2.3.2.** The condition on the characteristic of coefficients may seem strange since one does not know much about $T_{y(x)}(r)$ apriori. Once simple example is if we require $a_i, b_i, \tilde{a}_i, \tilde{b}_i$’s to be rational in $x$. Then, we know that their characteristic is $O(\log r)$ (see theorem 1.3.2), and since $y(x)$ is assumed to be transcendental meromorphic, the condition is automatically satisfied (see theorem 1.3.3).
2.3. THE LEMMA ON LOGARITHMIC DERIVATIVES REVISITED

Proof. Observe that by theorem 1.4.2,
\[ T(r, R(x, y^{(n)})) = T(r, R_1(x, y)) \]
\[ \Rightarrow dT(r, y^{(n)}) = d_1 T(r, y) + o(T(r, y)) \]
Applying (2.29) yields
\[ d(n + 1)T(r, y) + O(\log(rT(r, y))) \geq d_1 T(r, y) + o(T(r, y)) \] (2.57)
Dividing by \( T(r, y) \) and letting \( r \rightarrow \infty \) yields the desired result.

A particular case is the differential equation
\[ \left( \frac{dy}{dx} \right)^k = R_1(x, y). \] (2.58)
Where \( R_1(x, z) \) has numerator with degree \( p \) in \( z \) and denominator with degree \( q \) in \( z \). Suppose such an equation has a transcendental meromorphic solution, and make the change of variables
\[ Y = \frac{1}{y - \alpha}, \alpha \in \mathbb{C} \]
yields
\[ \left( \frac{dY}{dx} \right)^k = -Y^{2k} R_1 \left( x, \frac{1}{Y} + \alpha \right) := R_2(x, Y) \] (2.59)

It does not take much to see that there are two cases,

Case 1: if \( p - q - 2k \geq 0 \), then the degree of numerator of \( R_2 \) in \( Y \) is \( p \) and the degree of the denominator is \( p - 2m \), which by application of the previous theorem yields
\[ 2m \geq p \geq q + 2k \Rightarrow q = 0, p = 2k. \]

Case 2: if \( p - q - 2k < 0 \), then the degree of the numerator of \( R_2 \) is \( q + 2k \), while the degree of the denominator is \( q \) and the previous theorem yields
\[ 2k \geq q + 2k > p \Rightarrow q = 0, p < 2k. \]

Put together, this means that \( R_1(x, y) \) is a polynomial in \( y \) with degree at most \( 2k \).

Remark 2.3.3. A special case is the case \( k = 1 \), where any equations of the form (2.58) reduce to a Riccati equation. Phrased differently, if a differential equation that is not a Riccati equation and is of the form (2.58) possesses meromorphic solutions, then such solutions are rational.

Another application establishes the following basic result

**Theorem 2.3.4.** Every meromorphic solution of the first Painlevé equation (PI)
\[ y'' = 6y^2 + z \]
possesses infinitely many poles.

Proof. To begin with, observe that PI does not possess any rational solutions. To see this, suppose to the contrary that a rational solution exists. Immediately, we see that if a function solves PI, it cannot be constant (plug a constant in). In fact, plugging in the ansatz \( y = c + O(z^{-1}) \) and comparing leading terms yields a contradiction. Hence, we must have that \( y = cz^\mu + O(z^{\mu-1}), \mu \in \mathbb{N} \). Plugging this in yields
\[ \mu(\mu - 1)z^{\mu-2} + O(z^{\mu-3}) = 6c^2 z^{2\mu} + O(z^{2\mu-1}) + z \]
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Now, if \( \mu \neq 1 \) and \( z^{\mu - 2}, z^{2\mu} \) are leading terms, then we get that \( \mu = -2 \), which is a contradiction because \( z \) would be the leading term on the RHS. If \( z^{\mu - 2}, z \) are the two leading terms, then \( \mu = 3 \), but that also yields a contradiction since \( z^{2\mu} \) would be the leading term of the RHS. One last option would be for \( z \) to cancel \( 6c^2z^{2\mu} \), but that would imply \( \mu = 1/2 \), contradicting rationality of \( y \).

Finally, in the case \( \mu = 1 \), we have
\[
O(z^{-2}) = 6c^2z^2 + O(z)
\]
an obvious contradiction. Hence, PI does not have any rational solutions.

Next, suppose \( y \) is a transcendental meromorphic solution of PI with finitely many poles and write
\[
y^2 = \frac{1}{6} \left( y \cdot \frac{y''}{y'} \cdot \frac{y'}{y} - z \right)
\]
then (here we use \( m_y(r, \infty) \equiv m(r, y) \))
\[
2m(r, y) = m(r, y^2) = m \left( r, \frac{1}{6} \left( y \cdot \frac{y''}{y'} \cdot \frac{y'}{y} - z \right) \right)
\]
\[
\leq m(r, 1/6) + m(r, y) + m \left( r, \frac{y''}{y'} \right) + m \left( r, \frac{y'}{y} \right)
\]
\[
\leq m(r, y) + O(\log r) + O(\log(rT_y(r)))
\]
Since \( 0 \leq m(r, y) \), we have established that \( m(r, y) = O(\log(rT_y(r))) \). Furthermore, since \( y \) has finitely many poles, \( N_y(r, \infty) = O(\log r) \). Hence, we have shown that
\[
T_y(r) = N_y(r, \infty) + m(r, y) = O(\log(rT_y(r)))
\]
dividing by \( T_y(r) \) and taking the limit as \( r \to \infty \) (keep in mind theorem 1.3.4) yields an obvious contradiction. \( \square \)

2.4 Application: Value Distribution

In the previous chapter, we proved the SMT and then focused on applications of the lemma on logarithmic derivatives. Here, we back-track to explore some of the consequences of the SMT. For convenience, we recall the statement here:

For any \( \{a_j\}_{j=1}^q \) distinct points in \( \hat{\mathbb{C}} \), there is a subset \( E \subset (0, \infty) \) of finite measure so that as \( r \to \infty \), \( r \notin E \),
\[
\sum_{j=1}^q m_f(r, a_j) + N_{1,f}(r) \leq 2T_f(r) + S(r)
\]
where \( S(r) = O(\log(rT_f(r))) \).

**Definition 2.4.1.** The deficiency of a value \( a \), \( \delta_f(a) \), is the limit
\[
\delta_f(a) := \liminf_{r \to \infty} \frac{m_f(r, a)}{T_f(r)} = 1 - \limsup_{r \to \infty} \frac{N_f(r, a)}{T_f(r)}
\]
while the critical deficiency is defined to be
\[
\Theta_f(a) := 1 - \limsup_{r \to \infty} \frac{N_f(r, a)}{T_f(r)}
\]
where \( N_f(r, a) \) counts the solutions of \( f(z) = a \) not counting multiplicity.
2.4. APPLICATION: VALUE DISTRIBUTION

The deficiency relation should indicate when a function takes on a value a "less than average" amount of times. Observe that if a value \( a \) is never assumed, then \( \delta_f(a) = 1 \). We will refer to a value \( a \in C \) as **deficient** if \( \delta(a) > 0 \).

**Remark 2.4.2.** Different notions of deficiency exist. One such notion is

\[
\Delta(a) := \limsup_{r \to \infty} \frac{m_f(r,a)}{T_f(r)} = 1 - \liminf_{r \to \infty} \frac{N_f(r,a)}{T_f(r)}.
\]

Here is the first consequence of the SMT

**Theorem 2.4.3.** Let \( f \) be a function meromorphic in \( C \). We have for any \( \{a_i\}_{i=1}^q \)

\[
\sum_{i=1}^q \delta_f(a_i) \leq \sum_{i=1}^q \Theta_f(a_i) \leq 2
\]

**Proof.** From the definition, \( N_f(a) \leq N_f(a) \implies \Theta_f(a) \geq \delta_f(a) \) and the first inequality follows. Next, observe that the difference \( N_f(r,a) - N_f(r,a) \) counts critical points satisfying \( f(z) = a \), and so,

\[
\sum_{i=1}^q (N_f(r,a) - N_f(r,a)) \leq N_{1,f}(r) \quad (2.61)
\]

Using (2.61) and the SMT yields

\[
\sum_{i=1}^q m_f(r,a_i) + \sum_{i=1}^q (N_f(r,a) - N_f(r,a)) \leq 2T_f(r) + S(r)
\]

Finally, using the FMT,

\[
qT_f(r) - \sum_{i=1}^q N_f(a_i) + O(1) \leq 2T_f(r) + S(r).
\]

Dividing through by \( T_f(r) \), taking a limit and recalling the definition of \( \Theta_f(a) \) yields

\[
\sum_{i=1}^q \Theta_f(a_i) \leq 2.
\]

**Corollary 2.4.4.** The set of deficient values of a function meromorphic in \( C \) is at most countable.

**Proof.** Let \( N = \{a \in C \mid \delta_f(a) > 0\} \), then

\[
N = \bigcup_{k=1}^{\infty} \left\{ a \mid \delta_f(a) > \frac{1}{k} \right\}
\]

and by the previous claim, each of the sets on the RHS is finite (otherwise, the summation in theorem 2.4.3 would diverge).

**Corollary 2.4.5** (Picard’s Theorem). An non-rational function meromorphic in \( C \) takes every value with at most two exceptions infinitely often.
Proof. If a value \( a \) is assumed only finitely many times, then \( N_f(r, a) \sim \log r \) and since \( f \) is assumed non-rational (i.e. \( \lim_{r \to \infty} T_f(r)/\log r = \infty \)), we have that \( \delta_f(a) = 1 \). Suppose 3 values \( a_1, a_2, a_3 \) are assumed only finitely many times. Then,

\[
\sum_{i=1}^{3} \delta_f(a_i) = 3 \geq 2
\]

contradicting theorem 2.4.3.

Remark 2.4.6. Although labeled Picard’s theorem, this is an intermediate between Picard’s little theorem, which states that a function meromorphic in \( \mathbb{C} \) attains all but two complex values, and Picard’s great theorem, which says that a function meromorphic in a neighborhood of infinity attains all but two values infinitely often in a neighborhood of infinity.

The above discussion shows that if a function \( f \) meromorphic in \( \mathbb{C} \) has two Picard values, then the deficiency attains its maximum. What is not so clear is that one can have any number of deficient points.

Example 2.4.6.1. Consider the function

\[
f(z) = \int_0^z e^{-t^p} \, dt. \quad (2.62)
\]

The convergence/divergence of the integral as \(|t| \to \infty\) along rays is determined via the sectors defined by

\[
W_\nu := \left\{ t : \left| \arg t - \frac{\nu \pi}{p} \right| < \frac{\mu}{2\pi} \right\}, \quad \nu = 0, 1, 2, ..., 2p - 1
\]

(sectors where \( \text{Re} \left( t^p \right) \) changes signs) where for odd values of \( \nu \), the integral diverges (\( \text{Re} \left( t^p \right) < 0 \)) and for even values of \( \nu = 2\mu \), the integral converges to

\[
a_\mu := e^{2\nu \pi / p} \int_{0}^{\infty} e^{-t^p} \, dt, \quad \mu = 0, 1, ..., p - 1
\]

Note that this is independent of which ray you consider in a given sector; to see this connect the two rays with the arc of a circle or radius \( r \) and see that for \( r \to \infty \), the integral along the arc vanishes). Consider \( W_{2\mu-1}, \mu = 1, 2, ..., p \). To write asymptotics for this function, one would use the standard trick, integration by parts. Consider any \( \epsilon > 0 \) such that \( \epsilon \to 0 \) as \( z \to \infty \).

\[
\int_{0}^{\infty} e^{-t^p} \, dt = \int_{0}^{\epsilon} e^{-t^p} \, dt + \int_{\epsilon}^{\infty} \frac{pt^{p-1}}{p}e^{-t^p} \, dt
\]

\[
= \int_{0}^{\epsilon} e^{-t^p} \, dt + e^{-\epsilon^p} - e^{-\epsilon^p} \frac{p - 1}{p} \int_{\epsilon}^{\infty} e^{-t^p} \, dt
\]

\[
= \frac{e^{-\epsilon^p}}{p^{2p-1}} \left( 1 + p^{p-1} \epsilon^p \int_{0}^{\epsilon} e^{-t^p} \, dt - \frac{e^{-\epsilon^p}}{p^{2p-1}} p^{p-1} e^{\epsilon^p} - p^{p-1} e^{\epsilon^p} \int_{\epsilon}^{\infty} e^{-t^p} \, dt \right).
\]

Noting that since, in \( W_{2\mu+1} \), \( \text{Re} \left( t^p \right) < 0 \), and if we consider \( \epsilon \to 0 \) as \( z \to \infty \), we have

\[
\int_{0}^{\infty} e^{-t^p} \, dt = \frac{e^{-\epsilon^p}}{p^{2p-1}} (1 + o(1)), \quad z \to \infty, \quad z \in W_{2\mu+1}
\]

(2.63)

This implies that for \( z = re^{\theta} \) (recall that \( \log \, |f| = \max \{0, |f| \} \))

\[
\log |f(z)| = -r^p \cos(p\theta) + o(1) \quad \Rightarrow \quad m(r, \infty) = \frac{p p^p}{2\pi} \int_{\pi/2p}^{3\pi/2p} \cos(p\theta) \, d\theta = \frac{r^p}{\pi} + o(1).
\]
Furthermore, \( N(r, \infty) \equiv 0 \) (\( f(z) \) is entire), which yields

\[
T_f(r) = \frac{r^p}{\pi} + o(1).
\]

Next, suppose \( z \in W_0 \) (i.e. \( \mu = 0 \)), then the integral converges and integration by parts yields

\[
\int_0^\infty e^{-zp} \, dt - \int_0^\infty e^{-r^p} \, dr = - \int_0^\infty e^{-t^p} \, dt
\]

\[
= - \frac{e^{-z^p}}{p z^{p-1}} + \frac{p - 1}{p} \int_z^\infty \frac{e^{-t^p}}{t^p} \, dt = - \frac{e^{-z^p}}{p z^{p-1}} (1 + o(1))
\]

Consequently,

\[
\log |f(z) - a_0| = -r^p \cos(p\theta) \implies m(r, a_0) = \frac{r^p}{p\pi} + o(1).
\]

In a similar way, we have

\[
m(r, a_\mu) = \frac{r^p}{p\pi} + o(1)
\]

and so, for \( \mu = 1, 2, \ldots, p \)

\[
\delta(a_\mu) = \frac{1}{p}, \quad \delta(\infty) = 1.
\]
Bibliography


