THE REAL TEAPOT

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Abstract. In his last paper, William Thurston defined the Master Teapot as the closure of the set of pairs (z, s), where s is the slope of a tent map T_s with the turning point periodic, and the complex number z is a Galois conjugate of s. In this case $1/z$ is a zero of the kneading determinant of T_s. We remove the restriction that the turning point is periodic, and sometimes look beyond tent maps. However, we restrict our attention to zeros $x = 1/z$ in the real interval $(0, 1)$. By the results of Milnor and Thurston, the kneading determinant has such a zero if and only if the map has positive topological entropy. We show that the first (smallest) zero is simple, but among other zeros there may be multiple ones. We describe a class of unimodal maps, so-called R-even ones, whose kneading determinant has only one zero in $(0, 1)$. In contrast with this, we show that generic mixing tent maps have kneading determinants with infinitely many zeros in $(0, 1)$. We prove that the second zero in $(0, 1)$ of the kneading determinant of a unimodal map, provided it exists, is always larger than or equal to $\sqrt[3]{1/2}$ and if the kneading sequence begins with $RL^N R$, $N \geq 2$, then the best lower bound for the second zero is in fact $\sqrt[N+1]{1/2}$. We also investigate (partially numerically) the shape of the *Real Teapot*, consisting of the pairs (s, x), where x in $(0, 1)$ is a zero of the kneading determinant of T_s, and $s \in (1, 2]$.

1. Introduction

1.1. **Master Teapot.** In his last paper [Th], William Thurston defined the Master Teapot as the closure of the set of pairs (z, s), where $s \in (1, 2]$ is the slope of a tent map T_s with the turning point periodic, and $z \in \mathbb{C}$ is a Galois conjugate of s (in fact the name Master Teapot does not appear in [Th] but, according to [BDLW], it was assigned to this object in Thurston's 2012 course at Cornell University).

Let us recall what a Galois conjugate is. If the real number s is transcendental, the Galois conjugates of s are not defined. If s is algebraic (i.e. a zero of a polynomial with rational, equivalently integer, coefficients), we take the *minimal polynomial* of s, i.e. the monic polynomial (polynomial with leading coefficient 1) with rational coefficients

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Date: October 24, 2023.

2020 *Mathematics Subject Classification.* Primary 37E05.

Key words and phrases. Kneading determinant, Master Teapot, Real Teapot, tent map, unimodal map.

Lluís Alsedà has been supported by the Spanish Ministerio de Ciencia e Innovación grant number PID2020-118281GB-C31 and from the Spanish Ministerio de Economía y Competitividad grant number MDM-2014-0445 within the "María de Maeztu" excellence program, and the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M). Jozef Bobok was supported by the European Regional Development Fund, project No. CZ 02.1.01/0.0/0.0/16_019/0000778. Michał Misiurewicz was supported by grant number 426602 from the Simons Foundation. Ľubomír Snoha was supported by VEGA grant 1/0158/20.

and with smallest degree, such that s is a zero of this polynomial. Then all the complex zeros of this polynomial are Galois conjugates of s.

Since in the definition of the Master Teapot it is assumed that the turning point of T_s is periodic, the map is Markov [ALM, p. 251]. The slope s is, by [ALM, Corollary 4.3.13], the exponential of the topological entropy. Hence, by [MT, Theorem 6.3], $1/s$ is the smallest positive zero of the kneading determinant D_s of T_s. Moreover, if $z \in \mathbb{C}$ is a Galois conjugate of s, then also $1/z$ is a zero of the kneading determinant of T_s – see formula (4). This means that one way to investigate the Master Teapot is to look at the zeros of the kneading determinants. We restrict our attention to real positive zeros of D_s. In particular, since the radius of convergence of D_s is 1, we look at the zeros in $(0, 1)$. On the other hand, we do not assume that the turning point of T_s is periodic, so we look at all $s \in (1, 2]$.

Thus, we define the *Real Teapot* as the set

$$\{(s, x) : D_s(x) = 0,\ x \in (0, 1),\ s \in (1, 2]\} \tag{1}$$

(note that we do not take the closure), see Figure 1. Consequently, we will concentrate on tent maps (and often only on topologically mixing ones, that is, those with slopes in $(\sqrt{2}, 2]$). However, whenever the proofs do not really require the unimodal maps to be tent, we will consider general unimodal maps.

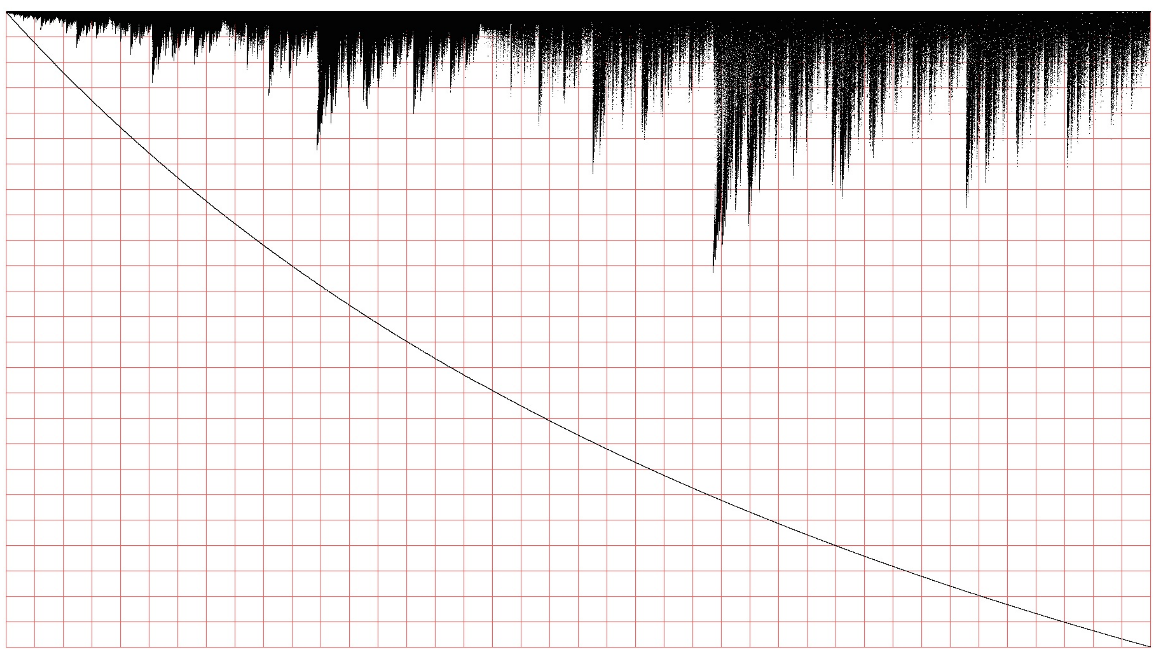


FIGURE 1. The Real Teapot. The horizontal axis is s from 1 to 2; the vertical axis is x, from 1/2 to 1.

Our Real Teapot bears some resemblance to the set $\widehat{\Sigma}$ from [T]. However, we are considering the family of all (not only Markov) tent maps with positive topological entropy instead of quadratic maps, and we are looking only at the zeros in the real interval $(0, 1)$ instead of from the unit disk. This makes our set considerably smaller, so we lose connectivity and perhaps local connectivity. Thus, considering the family of quadratic maps (or all unimodal maps) could allow us to get stronger results in those directions, but by (5) it would lead us further from the original ideas of Thurston, by introducing too many "non-Galois" points corresponding to the factor $D_B(x^n)$ in (5).

1.2. **Unimodal maps, kneading sequences and kneading determinants, tent maps.** By a *unimodal* map we will mean a continuous map $f\colon [u,v] \to [u,v]$, strictly increasing on $[u,c]$ and strictly decreasing on $[c,v]$ for some $c \in (u,v)$. Sometimes we encounter maps that are strictly decreasing on $[u,c]$ and strictly increasing on $[c,v]$ (which we also should call unimodal); then we conjugate them via an orientation reversing homeomorphism and they become unimodal in the previous sense.

Notice that

(i) if $f(c) \le c$ then $f([u,v]) \subset [u,f(c)] \subset [u,c]$, $f|_{[u,c]}$ is increasing, and
(ii) if $c \le f^2(c) < f(c)$ then $f([c,f(c)]) \subset [c,f(c)]$, $f|_{[c,f(c)]}$ is decreasing.

In both cases (i),(ii) $h(f) = 0$. In what follows we will be interested in maps with positive topological entropy, so we will consider only unimodal maps with

$$f^2(c) < c < f(c). \tag{2}$$

Then for every $x \in [f(c),v]$ one has $f(x) \in [u,f^2(c)]$ and all trajectories of points from $[u,f^2(c)]$ are monotone or eventually enter the interval $[f^2(c),f(c)]$. Since $f([f^2(c),f(c)]) \subset [f^2(c),f(c)]$, the nontrivial dynamics induced by a unimodal map f with positive topological entropy lives in $[f^2(c),f(c)]$ only. The interval $[f^2(c),f(c)]$ is usually called the *core of* f.

If we use an appropriate increasing linear map as a conjugacy, a unimodal map restricted to its core can always be considered on the interval $[0,1]$; then 1 is mapped to 0.

When we speak of unimodal maps, most of the time we consider them on their cores and we assume that the core is always $[0,1]$.

To get the *kneading (sequence)* $K = K_f$ of a unimodal map $f\colon [0,1] \to [0,1]$, we code the trajectory of 1^- using the symbols $R = (c,1]$ and $L = [0,c)$ (i.e. we code the trajectory of 1 but if the turning point c is periodic, we replace 1 by a slightly smaller point and we take the limit as this point which is coded goes to 1 from the left). Then we recode for a sequence of signs: start with $+$, R changes the sign, L does not; then the *kneading determinant* is the power series $D = D_f$ with the corresponding coefficients $+1$ and -1.

Thus, the kneading K always starts with RL and so the kneading determinant

$$D(x) = \sum_{k=0}^{\infty} \varepsilon_k x^k$$

starts with $D(x) = 1 - x - x^2 \dots$. Each ε_k is equal to $+1$ or -1, depending on whether f^k is increasing or decreasing, respectively, at the point 1^-. If the block of first k symbols in K contains odd or even number of R's, we have $\varepsilon_k = -1$ or $\varepsilon_k = +1$, respectively. For an example, see Figure 2.

In the important case when the turning point c is periodic with period n we can also write the kneading sequence in the form of a finite sequence $RLS_3 \cdots S_{n-1}C$, where $S_i \in \{L,R\}$. In this case, i.e. when $f^n(c) = c$, by [MT, Lemma 4.5, p. 486], the nth term in the kneading determinant is always $+x^n$ and not $-x^n$. The next term is

$-x^{n+1}$, since the turning point c is mapped to 1 and points in any small neighborhood of 1 are coded by R. Then the kneading determinant $1 - x - x^2 ... + x^n - x^{n+1} ...$ is a rational function with the denominator $1 - x^n$.

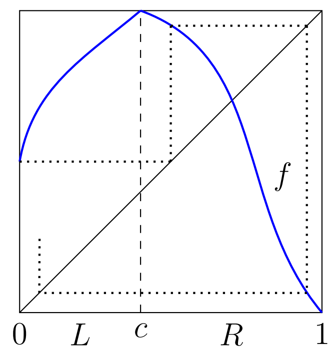


FIGURE 2. A unimodal map with the kneading sequence $K = RLRRL\dots$, the corresponding sequence of signs $+--+--\dots$ and the kneading determinant $D(x) = 1 - x - x^2 + x^3 - x^4 - x^5 \dots$.

In the set of all kneading sequences we have a natural order which takes into account the fact that the map is orientation preserving on the left lap and orientation reversing on the right one. Namely, if we have two distinct kneading sequences, we look for the first place where they differ. Then, if the number of symbols R in the common part is even, the ordering of the kneading sequences is the one induced by $L < C < R$, and if this number is odd, it is the one induced by $R < C < L$.

We will use the *Collet-Eckmann condition*, which tells us which sequences can be kneading sequences (see [CE, p. 71] where such sequences are called *maximal*). It tells us that $K = A_1 A_2 \dots A_n A_{n+1} \dots$ is a kneading sequence of a unimodal map if and only if K is greater than or equal to every shift of K, i.e., $K \geq A_n A_{n+1} \dots$ for every n. Those inequalities make sense since in the definition of the order we do not have to assume that the sequences involved are kneading sequences of unimodal maps.

In particular, it follows that if a kneading sequence starts with $RL^N R$, then it does not have any block of symbols L longer than N. This implies that in its sequence of signs there are no blocks of the same symbol longer than $N + 1$.

According to [CCE, Lemma 2], for unimodal maps topological entropy is an increasing function of the kneading sequence. We denote this function by H. Later, in Theorem 2.4, we will show some conditions for the points at which it is strictly increasing.

A particular example of a unimodal map is the map T_s, $s \in (1, 2]$ restricted to its core; as explained above we consider it rescaled to $[0, 1]$. The map T_s understood in this way, has constant slope s (in absolute value), the turning point $c = 1 - 1/s$ and $T_s(1) = 0$. Instead of K_{T_s} and D_{T_s} we use also notation K_s and D_s, respectively. Recall that T_s is mixing if and only if $T_s(0) < a$ where a is the fixed point in the decreasing lap, i.e. $a = s/(1 + s)$. This is equivalent to the condition $s \in (\sqrt{2}, 2]$.

1.3. **Galois conjugates of the slope. Zeros of the kneading determinant.** If α is an algebraic number, denote by $\mathrm{Gal}(\alpha)$ the set of Galois conjugates of α, i.e. the set of zeros of the minimal polynomial of α. The minimal polynomial of α is irreducible and conversely, if a monic polynomial with rational coefficients is irreducible and has α as a zero, then it is the minimal polynomial of α [H, Theorem 11.6].

Let $s \in (1, 2]$ be the slope of a tent map T_s. As explained above, $1/s$ is the smallest positive zero of the kneading determinant D_s. Assume that s is such that the turning point of T_s is periodic, which is an assumption in the definition of the Master Teapot. Then s is an eigenvalue of the transition matrix associated to the orbit of the turning point of T_s [ALM, Th. 4.4.5], so the slope s is an algebraic number.

Recall the following facts which hold true for the unimodal maps (restricted to the core and rescaled to $[0, 1]$).

According to [MT, Theorem 9.1], the reciprocal of the zeta function for a unimodal map f, when the turning point is periodic of period p, is $(1 - x^p)D(x)$. At the same time, by [BL, Theorem 1], for the subshift of finite type with the transition matrix M, the reciprocal of the zeta function is $\det(I - xM)$. Thus, $(1 - x^p)D(x) = \det(I - xM)$. Therefore,

$$D(x) = \frac{\det(I - xM)}{1 - x^p} = \frac{x^{p-1} \det\left(\frac{1}{x}I - M\right)}{1 - x^p}, \tag{3}$$

so the zeros of $D(x)$ are reciprocal to the eigenvalues of M.

In particular, formula (3) implies that

$$\left\{\frac{1}{z} \colon z \in \mathrm{Gal}(s)\right\} \subset \{z \in \mathbb{C} \colon D_s(z) = 0\}. \tag{4}$$

In Subsection 7.5 we will show an example where the opposite inclusion does not hold. Note that $\{1/z \colon z \in \mathrm{Gal}(s)\} = \mathrm{Gal}(1/s)$, which is a consequence of the fact that if a polynomial is irreducible then also the polynomial with the same coefficients but written in the reversed order, is irreducible.

If f is a unimodal map, its kneading determinant D is a function defined in the open unit disk in the complex plane and, being analytic, has only isolated zeros. We will study it (and still denote by D) in the real interval $[0, 1)$, so for any $\varepsilon \in (0, 1)$ the interval $(0, 1 - \varepsilon]$ contains only finitely many zeros of D. In particular, if the set of positive zeros of D is nonempty, then D has the first (smallest) zero that we will denote as $\mathrm{FZ}(D)$.

(F1) By [MT, Theorem 6.3], the kneading determinant D has a zero in the open interval $(0, 1)$ if and only if the topological entropy $h(f) > 0$. In such a case

$$h(f) = \log \frac{1}{\mathrm{FZ}(D)} \quad \text{and } D \text{ has no (complex) zeros in } |z| < \mathrm{FZ}(D).$$

Since $h(f) \leq \log 2$, we can see that if $h(f) > 0$ then $\mathrm{FZ}(D) \geq 1/2$.

(F2) Clearly $D(x) > 0$ for $0 \leq x < \mathrm{FZ}(D)$ and so, by [Pr, p.197, (1.17)],

$$D'(x) < 0 \quad \text{for } 0 \leq x < \mathrm{FZ}(D).$$

If D has no positive zero, i.e. if $h(f) = 0$, we have both $D(x) > 0$ and $D'(x) < 0$ for all $0 \leq x < 1$.

In the present paper we want to obtain more results on the zeros of the kneading determinants of unimodal maps. To illustrate the complexity of the problem, look at the maps T_s, $s \in (1, 2]$. They have positive entropy $\log s$ and, by (F1), the first zero of the kneading determinant for them is $1/s$, a continuous function of the slope s. Unlike the first zero, the situation with the second and higher zeros is more complicated. Figure 3 shows all zeros except the first ones (the curve $1/s$ showing the first zeros would be below the visible part of the figure). Vertical red lines are every 0.0015, and horizontal ones every 0.01 (so the highest horizontal line is at $x = 1$ and the lowest one at $x = 0.79$). Every horizontal pixel represents 1000 values of s. By computational reasons, what we see in the highest row of the squares is not reliable; see also Section 5. The set of the points plotted in the figure is the essential part of the Real Teapot. The rest of it lies to the left or below the area shown in the figure.

When speaking of zeros of a kneading determinant, we will have only positive zeros, i.e. zeros in $(0, 1)$, in mind. In particular, the *first zero* or the *second zero* will mean the smallest zero or the second smallest zero in $(0, 1)$, respectively.

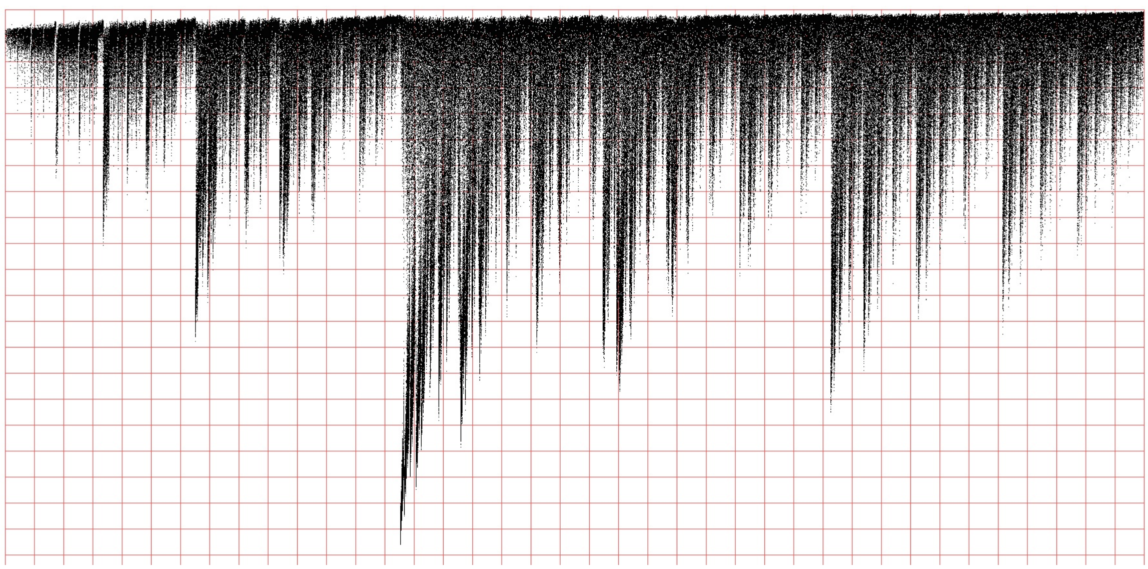


FIGURE 3. All zeros except the first ones, for the slopes $s \in [\sqrt{2}, 2]$. It resembles Spanish Moss (Tillandsia usneoides), so maybe this should be the name of our set, rather than Real Teapot?

1.4. **Renormalization and star product.** Many unimodal maps are *renormalizable*. This means that there is an invariant cycle of intervals with disjoint interiors, one of those interiors containing the turning point (if the period of that cycle is n, we speak on n-renormalizability). In that case the kneading sequence of the map is a *star product* $K = \underline{A} * B$, where $\underline{A}C$ is a finite kneading sequence, and B is an arbitrary kneading sequence.

The map obtained by collapsing all intervals of the cycle, and their preimages, to points, has kneading sequence $\underline{A}C$. The map of the first return to an interval of the cycle (we can choose any of them; it is called a *restrictive interval*), restricted to this interval, has kneading sequence B.

Remember that it may happen that this map is again renormalizable.

In fact, the star product can be even applied to sequences that do not satisfy the Collet-Eckmann condition. We send readers that want to find the formal definition of it to [CE]. Here we will need only to know two specific cases, which will be used in the proof of Lemma 4.5. Namely, $RL * RL^\infty = RLL(RLR)^\infty$, and $RL * L^\infty = (RLR)^\infty$.

Two things important to us are that topologically mixing tent maps are not renormalizable, and that there is a nice formula for the kneading determinant of renormalizable maps. Namely, if the kneading determinants of maps with kneading sequences $K, \underline{A}C, B$ are D_K, D_A, D_B respectively, $K = \underline{A} * B$, and the length of $\underline{A}C$ is n, then, according to [MT], equation (14.9), we have

$$D_K(x) = (1 - x^n) D_A(x) D_B(x^n). \tag{5}$$

Observe that if $n \geq 3$, then the zeros contributed by $D_B(x^n)$ are not smaller than $\sqrt[3]{1/2}$ (compare Subsection 1.5 below).

Related to this, it is well known that if $K = \underline{A} * B$ and the length of $\underline{A}C$ is larger than 2 then the topological entropies of unimodal maps with the kneading sequences K and $\underline{A}C$ are equal. However, if the length of $\underline{A}C$ is 2 (that is, $\underline{A} = R$; this corresponds to a decreasing map, i.e. $c = 0$), then the topological entropy of a map with the kneading sequence K is one half of the topological entropy of a map with the kneading sequence B, because in this case $D_R(x) = 1/(1 + x)$. So, formula (5) yields

$$D_K(x) = (1 - x) D_B(x^2). \tag{6}$$

Observe that this case corresponds to a 2-renormalization.

To illustrate it, consider the tent map T_2. Its kneading sequence is $K_2 = RL^\infty$ and the kneading determinant $D_2(x) = 1 - x - x^2 - x^3 - \ldots = (1 - 2x)/(1 - x)$. If we want to find the kneading determinant for $T_{\sqrt{2}}$, we can proceed directly by realizing that $K_{\sqrt{2}} = RLR^\infty$ and so $D_{\sqrt{2}}(x) = 1 - x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots = (1 - 2x^2)/(1 + x)$, or we can use formula (6). Indeed, $T_{\sqrt{2}}$ is 2-renormalizable; $[0, a]$ and $[a, 1]$, where $a = 2 - \sqrt{2}$ is the fixed pint of $T_{\sqrt{2}}$, form a 2-cycle of intervals. The first return map to $[a, 1]$, i.e. the restriction of $(T_{\sqrt{2}})^2$ to $[a, 1]$, is topologically conjugate to T_2. We therefore have $K_{\sqrt{2}} = R * K_2$ and by (6),

$$D_{\sqrt{2}}(x) = (1 - x) D_2(x^2) = (1 - x)\frac{1 - 2x^2}{1 - x^2} = \frac{1 - 2x^2}{1 + x}.$$

Similarly, one can get $D_{\sqrt[2^{k+1}]{2}}(x)$ from $D_{\sqrt[2^k]{2}}(x)$.

1.5. **Main results.**

We summarize our main results and describe the organization of the paper.

In Section 2 we describe a class of unimodal maps, so-called R-even ones, whose kneading determinant has just one zero in $(0, 1)$; see Theorem 2.8.

In contrast with this, in Section 3 we show that generically the mixing tent maps have kneading determinants with infinitely many zeros in $(0, 1)$; see Theorem 3.2.

In Section 4 we investigate how large the second zero in $(0, 1)$ of the kneading determinant of a unimodal map is, provided it exists. In Theorem 4.12 we show that it is always larger than or equal to $\sqrt[3]{1/2} \approx 0.793701$ (if the kneading sequence begins with $RL^N R$, $N \geq 2$, then the best lower bound for the second zero is in fact $\sqrt[N+1]{1/2}$, see Proposition 4.7). Moreover, in the same theorem we show that the first zero, if it exists, is always simple. In general this is not true for other zeros in $(0, 1)$; see Figures 12 and 13.

The values $s = \sqrt[2^k]{2}$, $k \geq 0$, belong to those for which the kneading determinant of the tent map T_s has only one zero in $(0,1)$, but both for $s \to 2^-$ and $s \to \sqrt[2^k]{2}$, $k \geq 1$, the limits of the second zeros are equal to 1, see Theorem 4.14.

In section 5 we consider the set valued function sending the slope s to the set of all zeros in $(0,1)$ of the kneading determinant of the tent map T_s. We look at a neighborhood of the golden mean s_3; this slope corresponds to the situation when the turning point is periodic of period 3. Some computations and arguments indicate that the graph of that set valued function, considered as a subset of the product of the neighborhood of s_3 with $(0,1)$ contains the union of countably many graphs of continuous functions with various connected domains. The pictures show a fractal structure.

In Section 6 we investigate what changes if we replace the Real Teapot by its closure.

Finally, in Section 7, we suggest some questions for possible further research.

2. When kneading determinant has just one zero in $(0,1)$

As we know, the kneading determinant of a unimodal map has a positive zero if and only if the map has positive topological entropy. When does it have only one positive zero?

We start with the following fact about power series.

Lemma 2.1. *Let $\varphi(x) = \sum_{n=0}^{\infty} \varepsilon_n x^n$, where $\varepsilon_n \in \{-1, +1\}$, $\varepsilon_0 = +1$, and if $\varepsilon_n = +1$ then $\varepsilon_{n+1} = -1$. Then $\varphi'(x) < 0$ for all $x \in (0,1)$.*

Proof. Define sequences $(\varepsilon_n^{(k)})_{n=0}^{\infty}$ by induction on k. First set $\varepsilon_n^{(1)} = +1$ if n is even and $\varepsilon_n^{(1)} = -1$ if n is odd. Suppose that $(\varepsilon_n^{(k)})_{n=0}^{\infty}$ is defined, let m be the smallest integer for which $\varepsilon_m^{(k)} \neq \varepsilon_m$, and suppose that $\varepsilon_m = -1$. Then set $\varepsilon_n^{(k+1)} = \varepsilon_n^{(k)}$ if $n < m$ and $\varepsilon_n^{(k+1)} = -\varepsilon_n^{(k)}$ if $n \geq m$. Observe that unless $(\varepsilon_n^{(k+1)})_{n=0}^{\infty} = (\varepsilon_n)_{n=0}^{\infty}$, if ℓ is the smallest integer for which $\varepsilon_\ell^{(k+1)} \neq \varepsilon_\ell$, then $\varepsilon_\ell = -1$, so we can continue induction.

Set $\varphi_k(x) = \sum_{n=0}^{\infty} \varepsilon_n^{(k)} x^n$ (if the induction stopped, then we just repeat the last φ_k). The functions φ_k (and φ) are analytic in the unit disk and the sequence (φ_k) converges to φ uniformly on compact sets. Therefore the sequence (φ_k') converges to φ' uniformly on compact sets.

We have $\varphi_1(x) = \frac{1}{1+x}$, so $\varphi_1'(x) = \frac{-1}{(1+x)^2} < 0$ on $(0,1)$. Moreover,

$$\varphi_{k+1}(x) - \varphi_k(x) = -\frac{2x^m}{1+x},$$

so

$$\varphi_{k+1}'(x) - \varphi_k'(x) = -2\,\frac{mx^{m-1} + (m-1)x^m}{(1+x)^2} < 0$$

on $(0,1)$. Therefore, (φ_k') is a decreasing sequence of negative functions on $(0,1)$, so its limit φ' is negative. □

As we know from Subsection 1.2, the kneading sequence of a unimodal map always starts with RL. We will call such a sequence *R-even* if except the leading R, all other Rs come in blocks of even lengths (that includes a possible infinite block). We will also call a unimodal interval map *R-even* if its kneading sequence is R-even. We are really interested only in unimodal maps with positive topological entropy (then the kneading determinant has at least one positive zero). In particular, in Corollary 2.6 we show that R-even unimodal maps do have positive topological entropy.

Recall that for a kneading sequence K we denoted by $H(K)$ topological entropy of maps with the kneading sequence K, and that H is increasing.

We are going to prove a theorem about kneading sequences at which H is strictly increasing. This theorem is well known, but we could not find its proof in the literature.

Let f be a unimodal map of positive topological entropy. By [MT], it is semiconjugate to a tent map T_f of the same entropy. The semiconjugacy φ_f is increasing and maps the right endpoint to the right endpoint (so it maps the turning point of f to the turning point of T_f). Thus, it maps the left (right) lap of f to the left (right) lap of T_f. Denote the turning point of f by c_f, and the turning point of T_f by c_{T_f}.

We consider first unimodal maps with topological entropy larger than $(1/2)\log 2$.

Lemma 2.2. *If the point c_{T_f} is periodic for T_f then either c_f is periodic for f or the kneading sequence of f is a star product.*

Proof. Assume that $T_f^n(c_{T_f}) = c_{T_f}$ for some $n > 0$. If $\varphi_f^{-1}(c_{T_f})$ consists of one point, then $f^n(c_f) = c_f$; otherwise it is an interval whose trajectory forms an invariant cycle of intervals, so the kneading sequence of f is a star product. □

Lemma 2.3. *If f and g have the same topological entropy larger than $(1/2)\log 2$, and different kneading sequences, then for each of f, g either the turning point is periodic, or the kneading sequence is a star product.*

Proof. Since $h(f) = h(g)$, we have $T_f = T_g$. Thus, for one of the maps (say, f), the kneading sequence is different than for T_f. Therefore, $\varphi_f(f^n(c_f)) = c_{T_f}$ for some $n > 0$. This implies that $T_f^n(c_{T_f}) = c_{T_f}$. Now we apply Lemma 2.2 to each of f, g. □

Now we replace the assumption that the topological entropy is larger than $(1/2)\log 2$ by a weaker assumption that it is positive. We will write $R^{*m} * B$ for $R * R * \dots * R * B$, where R is repeated m times.

Theorem 2.4. *Assume that K is the kneading sequence of a unimodal map with positive entropy, for which the turning point is not periodic, and K is not of the form $R^{*m} * B$ for $m \geq 0$ and a kneading sequence B which is a star product. Then H is strictly increasing at K.*

Proof. Let K be the kneading sequence of f. Assume first that $h(f) > (1/2)\log 2$. Then by Lemma 2.3, for any kneading sequence $K^* \neq K$ we have $H(K^*) \neq H(K)$.

Now, if $K = R^{*m} * B$ for a kneading sequence B which is not a star product, we reduce it to the former case by taking m 2-renormalizations. We do not have to consider the case when K and K^* are 2-renormalizable different number of times, because then clearly $H(K^*) \neq H(K)$.

Finally, we use the fact that H is increasing. □

Since we work with unimodal maps of positive topological entropy (increasing on the first lap and decreasing on the second lap, and restricted to the core, which is always assumed to be $[0, 1]$), the kneading sequence always starts with RL. Therefore, there are kneading sequences of the following four types: $RLL\dots$, RLR^∞,

$RLR^{2N}L\dots$ and $RLR^{2N-1}L\dots$ for $N \geq 1$. Moreover, we have

$$RLR^{2N-1}L\dots < RLR^{\infty} < RLR^{2N}L\dots < RLL\dots,$$

and so we immediately get the following corollary (recall that a is the fixed point in the decreasing lap).

Corollary 2.5. *The function H is strictly increasing at RLR^{∞}. In particular, if $f\colon [0,1] \to [0,1]$ is a unimodal map with positive topological entropy, then*

(a) $K_f = RLR^{\infty}$ *if and only if* $h(f) = (1/2)\log 2$.
(b) $K_f = RLL\dots$ *or* $K_f = RLR^{2N}L\dots$, $N \geq 1$, *if and only if the topological entropy $h(f)$ belongs to the interval $((1/2)\log 2, \log 2]$. In this case $f(0) < a$.*
(c) $K_f = RLR^{2N-1}L\dots$, $N \geq 1$, *if and only if $h(f) \in (0, (1/2)\log 2)$. In this case $f(0) > a$.*

Corollary 2.6. *All R-even unimodal maps have topological entropy in the interval $[(1/2)\log 2, \log 2]$.*

Lemma 2.7. *Let $\sum_{n=0}^{\infty} \varepsilon_n x^n$ be a kneading determinant and let K be the associated kneading sequence. Then, $\sum_{n=0}^{\infty} \varepsilon_n x^n$ satisfies the assumptions of Lemma 2.1 if and only if K is R-even.*

Proof. To compute the kneading determinant from the kneading sequence, we start with setting $\varepsilon_0 = +1$, and then we replace R by -1 and L by $+1$. Next, we take the products of initial parts to get our sequence (ε_n); recall that it starts with $+1, -1, -1$.

If the kneading sequence is R-even, then Rs come in pairs. Therefore, if $\varepsilon_n = +1$ for some $n \geq 1$, then $\varepsilon_{n+1} = -1$.

If there is an additional R at the end of a block of Rs, we will get $\varepsilon_n = \varepsilon_{n+1} = +1$. This completes the proof. □

From Lemmas 2.1 and 2.7 and Corollary 2.6 we get immediately the following theorem.

Theorem 2.8. *Assume that a unimodal map f is R-even. Then its kneading determinant has negative derivative and a unique zero in $(0, 1)$.*

As before, for a unimodal map $f\colon [0,1] \to [0,1]$ we denote the turning point by c and the fixed point (to the right of c) by a. We will also denote the prefixed point (the point other than a, whose image is a) by a', provided it exists. Obviously, it is the case if and only if $f(0) \leq a$.

Lemma 2.9. *A unimodal map f with $f(0) \leq a$ is R-even if and only if the trajectory of c misses the interval (a', c).*

Proof. If the trajectory of c misses the interval (a', c) and x belongs to this trajectory, then the assumptions $x < c$ and $f(x) > c$ imply $f(x) \leq a$. This produces blocks of Rs of even lengths. Similarly, if $x \in (a', c)$, then $f(x) > a$, and this produces a block of Rs of odd length. □

Further, we denote by b the point in the decreasing lap for which $f(b) = f(0)$. For the purposes of this paper, the trajectory (or the orbit) of the turning point of a unimodal map f with $f(0) \leq a$ (so a' exists) is said to be *twist* if it misses the intervals $[a', c)$ and $[a, b]$.[1] For an example of a twist orbit see Figure 4.

[1]One can show that our definition of a twist orbit is equivalent to the standard one from [B].

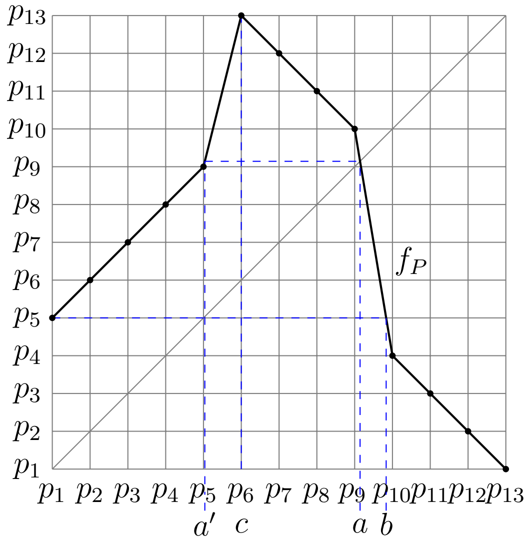


FIGURE 4. A twist orbit $P = \{0 = p_1 < p_2 < \cdots < p_{13} = 1\}$.

Combining Theorem 2.8 and Lemma 2.9 we obtain the following corollary.

Corollary 2.10. *Assume that the trajectory of the turning point of a unimodal map f is twist. Then its kneading determinant has negative derivative and a unique zero in* $(0,1)$*.*

Theorem 2.8 shows that R-evenness is a sufficient condition for the kneading determinant to have a unique zero in $(0,1)$. The condition is not necessary. Such an example exists even if the turning point is periodic.

Example 2.11 (One zero without kneading sequence being R-even)**.** For $n \in \mathbb{N}$, the tent map T_{s_n} with the kneading sequence $RLLR^{2n-1}C$ has the turning point periodic with period $2n+3$. By (F1), $x_n = 1/s_n \in (1/2, 1)$ is the first zero in $(0,1)$ of the kneading determinant of T_{s_n}. The kneading sequence $K_{s_n} = RLLR^{2n-1}C$ can be written also in the form $K_{s_n} = (RLLR^{2n-1}L)^\infty$. The beginning block $RLLR^{2n-1}$ yields the beginning block of the kneading determinant,

$$\begin{aligned} Q(x) &:= 1 - x - x^2 - (x^3 - x^4) - (x^5 - x^6) - \cdots - (x^{2n+1} - x^{2n+2}) \\ &= 1 - x - x^2 - x^3 \frac{1 - x^{2n}}{1+x} = \frac{x^{2n+3} - 2x^3 - 2x^2 + 1}{1+x} \ . \end{aligned}$$

Then the kneading determinant is

$$D_{s_n}(x) = Q(x) + x^{2n+3}Q(x) + (x^{2n+3})^2 Q(x) + \cdots = \frac{x^{2n+3} - 2x^3 - 2x^2 + 1}{(1+x)(1 - x^{2n+3})} \ .$$

To show that $D_{s_n}(x)$ has only the zero x_n in $(0,1)$, we look at the first three derivatives of the polynomial $P_n(x) = x^{2n+3} - 2x^3 - 2x^2 + 1$ (we think of $P_n(x)$ and its derivatives as defined only on $[0,1]$):

$$
\begin{aligned}
(P_n)'(x) &= (2n+3)x^{2n+2} - 6x^2 - 4x,\\
(P_n)''(x) &= (2n+3)(2n+2)x^{2n+1} - 12x - 4,\\
(P_n)'''(x) &= (2n+3)(2n+2)(2n+1)x^{2n} - 12.
\end{aligned}
$$

The third derivative is strictly increasing, with $(P_n)'''(0) < 0$ and, since $n \geq 1$, $(P_n)'''(1) > 0$. Hence, there is point $t_n \in (0,1)$ such that $(P_n)'''(x)$ is negative for $x \in [0,t_n)$ and positive for $x \in (t_n,1]$. It follows that $(P_n)''(x)$ is strictly decreasing on $[0,t_n]$ and strictly increasing on $[t_n,1]$. Since $(P_n)''(0) < 0$ and $(P_n)''(1) > 0$, there is a point $q_n \in (t_n,1)$ such that $(P_n)''(x)$ is negative for $x \in [0,q_n)$ and positive for $x \in (q_n,1]$. Thus $(P_n)'(x)$ is strictly decreasing on $[0,q_n)$ and strictly increasing on $(q_n,1]$. Since $(P_n)'(1) = 2n+3-10$, we distinguish two cases.

Case 1: $n \in \{1,2,3\}$. In this case $(P_n)'(0) = 0$ and $(P_n)'(1) < 0$. Hence, using the properties of $(P_n)'(x)$, we can see that $(P_n)'(x)$ is negative for $x \in (0,1]$. Therefore $(P_n)(x)$ is strictly decreasing on $[0,1]$ and so x_n is the only zero of P_n in $(0,1)$.

Case 2: $n \geq 4$. In this case $(P_n)'(0) = 0$ and $(P_n)'(1) > 0$. It follows that there is a point $r_n \in (q_n,1)$ such that $(P_n)'(x)$ is negative for $x \in (0,r_n)$ and positive for $x \in (r_n,1]$. Thus, $P_n(x)$ is strictly decreasing on $[0,r_n]$ and strictly increasing on $[r_n,1]$ Since $P_n(0) > 0$ and $P_n(1) < 0$, we can see that $P_n(r_n) < 0$ and $P_n(x)$ has exactly one zero, which lies in $(0,r_n)$ (it is of course x_n).

We have shown that in both cases $P_n(x)$ has just one zero in $(0,1)$, so $D_{s_n}(x)$ has just one zero in $(0,1)$.

We know from [B] that there are infinitely many slopes s for which the tent map T_s has the trajectory of the turning point periodic and twist. This gives us countably many values of s for which T_s is R-even. However, we can show that there are uncountably many such values.

Theorem 2.12. *There is a set $\mathscr{R} \subset (\sqrt{2},2)$ of cardinality of the continuum such that for every $s \in \mathscr{R}$ the trajectory of the turning point of T_s is twist and the turning point of T_s is not periodic.*

Proof. For $s \in [\sqrt{2},2]$ consider the points a, a', c, b defined as before. Additionally set $d = T_s(0)$. We conjugate T_s via a map $\psi : [0,1] \to [0,1]$, defined by $\psi(x) = x$ for $x \leq a$ and $\psi(x) = 1 + a - x$ for $x \geq a$ (we think of ψ having two values at a). The result is $g_s = \psi \circ T_s \circ \psi^{-1}$ (see Figure 5). We set $b' = \psi(b)$. Since ψ is discontinuous, so is g_s. Then we modify g_s to f_s by making it constant on the intervals $[a',c]$ (where we set f_s to have value a) and $[b',1]$ (where we define f_s to have value d); see Figure 5.

The map f_s can be considered as a continuous monotone circle map of degree 1. Then it has a rotation number, which depends continuously on the map [ALM, Lemma 3.7.12].

Since f_s depends continuously on s, this rotation number $\varrho(s)$ depends on s. Direct computations show that $\varrho(\sqrt{2}) = 1/2$ and $\rho(2) = 0$. Therefore there is a set $\mathscr{R} \subset (\sqrt{2},2)$ of cardinality of the continuum such that for each $s \in \mathscr{R}$ the rotation number $\varrho(s)$ is irrational, and therefore f_s has no periodic points.

Now fix $s \in \mathscr{R}$ and suppose that either the trajectory of the turning point of T_s is not twist, or the turning point of T_s is periodic. Then there is $n > 0$ such that

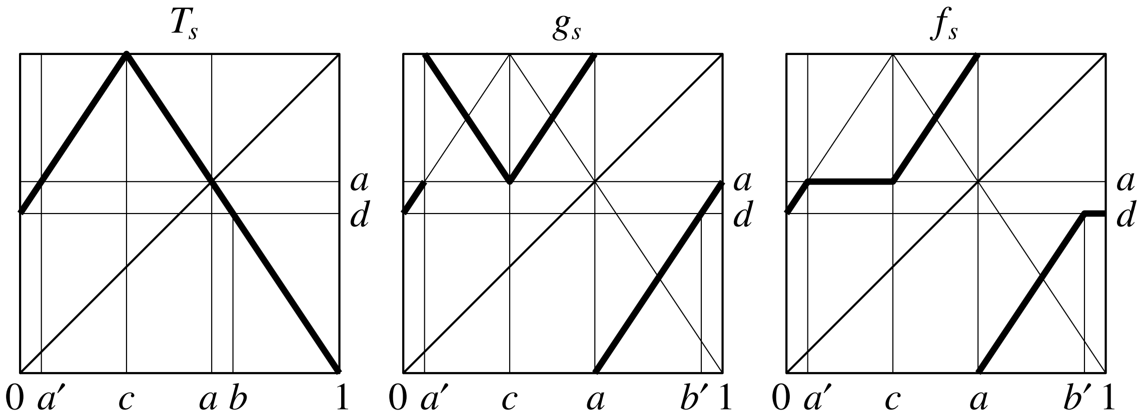


FIGURE 5. Construction of f_s.

$T_s(c), \dots, T_s^{n-1}(c)$ do not belong to $[a', c] \cup [a, b]$, but $T_s^n(c)$ does. Observe that $n \geq 3$. Further notice that the conjugacy ψ maps $[a', c] \cup [a, b]$ onto $[a', c] \cup [b', 1]$. Then $f_s^k(c) = g_s^k(c)$ for $k = 1, \dots, n$, and $f_s^k(c) \notin [a', c] \cup [b', 1]$ for $k = 1, \dots, n-1$, while $f_s^n(c) \in [a', c] \cup [b', 1]$. We have $f_s^3([a', c]) = d$ and $f_s([b', 1]) = d$. Since $f_s^3(c) = d$, we have $f_s^{n-3}(d) = f_s^n(c)$. Therefore, either $f_s^{n-3}(d) \in [a', c]$, and then $f_s^n(d) = d$, or $f_s^{n-3}(d) \in [b', 1]$, and then $f_s^{n-2}(d) = d$. In both cases we get a periodic point for f_s, so $s \notin \mathscr{R}$, a contradiction. This proves that if $s \in \mathscr{R}$ then the trajectory of the turning point of T_s is twist and the turning point of T_s is not periodic. $\square$

3. Real zeros of kneading determinants of generic tent maps

We show that generically the mixing tent maps have kneading determinants with infinitely many zeros in $(0, 1)$.

Let T_s be the tent map with slope s, where $\sqrt{2} < s \leq 2$. For any $n \geq 3$, set $v_n = (1/2)^{1/n}$.

Lemma 3.1. *If for T_s the turning point is periodic of period n, and $V \subset (0, 1)$ is a neighborhood of v_n, then there is an open interval $J \subset (\sqrt{2}, 2)$ having s as an endpoint, and such that for every $t \in J$ the kneading determinant of T_t has a zero in V.*

Proof. Let s be such that for T_s the turning point is periodic of period n. Suppose first that v_n is a zero of D_s of odd multiplicity. Then any function sufficiently close to D_s has a zero close to v_n. If $t < s$ is close enough to s, then the series for D_t coincides with the series for D_s up to some very large powers of x, so for $x \in (0, (v_n + 1)/2)$ the function D_t is close to D_s. Consequently, D_t has a zero in $V \cap (0, (v_n + 1)/2)$. This means that s is the right endpoint of some interval $J \subset (\sqrt{2}, 2)$, such that for every $t \in J$ the kneading determinant of T_t has a zero in V.

Suppose now that v_n is not a zero of D_s of odd multiplicity. If the kneading sequence of T_s is $\underline{A}C$, consider a unimodal map f with kneading sequence $\underline{A} * RL^\infty$. According to (5), we have

$$D_f(x) = (1 - x^n)\, D_s(x)\, D_2(x^n) = D_s(x)\, (1 - 2x^n).$$

The function $1 - 2x^n$ has a simple zero at v_n. By our assumption, v_n is not a zero of D_s of odd multiplicity, so it is a zero of $D_s(x) \cdot (1 - 2x^n)$ of odd multiplicity. If $t > s$ is close enough to s, then the series for D_t coincides with the series for D_f up to some very large powers of x. So by the same arguments as before, D_t has a zero in V. No tent map has a kneading sequence between K_s and K_f, so s is the left endpoint of some interval $J \subset (\sqrt{2}, 2)$, such that for every $t \in J$ the kneading determinant of T_t has a zero in V. $\square$

Now we can show that for a generic $s \in (\sqrt{2}, 2)$, the kneading determinant of T_s has infinitely many zeros in $(0, 1)$. In fact, we prove a slightly stronger theorem.

Theorem 3.2. *Let $U \subset (0, 1)$ be a neighborhood of the set $\{v_n : n \geq 3\}$. Then the set of those $s \in (\sqrt{2}, 2)$, for which D_s has infinitely many zeros in U, is residual in $(\sqrt{2}, 2)$.*

Proof. For every $N > 3$, the set of those $s \in (\sqrt{2}, 2)$ for which the turning point of T_s is periodic of period larger than N, is dense in $(\sqrt{2}, 2)$. Therefore, by Lemma 3.1, the set of those $t \in (\sqrt{2}, 2)$ for which D_t has a zero in $U \cap (v_N, 1)$, contains an open dense subset $Z_N \subset (\sqrt{2}, 2)$. The intersection $Z = \bigcap_{N=4}^{\infty} Z_N$ is residual and if $s \in Z$ then D_s has infinitely many zeros in U. $\square$

4. Lower bounds for the second zero of the kneading determinant

In this section we prove that the second positive zero of the kneading determinant of a unimodal map with positive entropy, if it exists, is greater than or equal to $\sqrt[3]{1/2} \approx 0.793701$.

Consider the family

$$\mathscr{D} := \{g \colon g \text{ is a differentiable function } [0, 1) \to \mathbb{R} \text{ and } g(0) = 1\}.$$

Our unimodal maps are defined on $[0, 1]$, with the first lap increasing and the second one decreasing, with the turning point mapped to 1 and 1 mapped to 0. Notice that the restriction of the kneading determinant of any unimodal map to the interval $[0, 1)$ belongs to $\mathscr{D}$.

The next lemma and its proof are illustrated in Figure 6.

Lemma 4.1. *Let $D, D^+ \in \mathscr{D}$ and let α, β, γ be such that $0 < \alpha < \gamma < \beta < 1$. Assume that*

(i) *$D(x) \leq D^+(x)$ for $x \in (0, \beta)$, with $D^+(x) < 0$ for $x \in (\alpha, \beta)$,*
(ii) *$D'(x) < 0$ for $x \in (0, \gamma)$.*

Then D has a unique zero in $(0, \alpha]$ (which, of course, is the first zero of D), and the function D has negative derivative at it. Moreover, the second positive zero of D, if it exists, is in $[\beta, 1)$.

Proof. Since $D(0) = 1$ and $D(\alpha) \leq D^+(\alpha) \leq 0$, the function D has a zero in $(0, \alpha]$. This zero is unique and D has negative derivative at it, since $D'(x) < 0$ for $x \in (0, \alpha]$. Suppose D has the second positive zero z. Then $z > \alpha$, and since $D(x) \leq D^+(x) < 0$ for $x \in (\alpha, \beta)$, we get $z \geq \beta$. $\square$

Typically, we will use the next lemma inductively to obtain a power series D^+ satisfying the assumptions of Lemma 4.1 from a power series D.

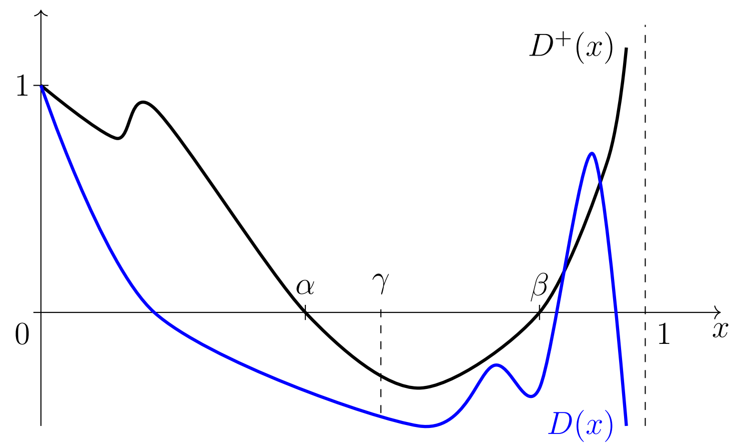


FIGURE 6. The functions D and D^+ from Lemma 4.1.

Lemma 4.2. *Let $D(x) = 1 + \sum_{k=1}^{\infty} \varepsilon_k x^k$ and $\widehat{D}(x) = 1 + \sum_{k=1}^{\infty} \delta_k x^k$, where $\varepsilon_k, \delta_k \in \{-1, 1\}$.*

1. *If $\delta_k \geq \varepsilon_k$ then $D(x) \leq \widehat{D}(x)$ and $D'(x) \leq \widehat{D}'(x)$ for every $x \in (0,1)$ (moreover, these two inequalities are strict provided the inequality $\delta_k \geq \varepsilon_k$ is strict for at least one k).*
2. *If there exists n such that $\varepsilon_n = -1$ and $\varepsilon_{n+1} = 1$, $\delta_k = -\varepsilon_k$ for $k = n, n+1$, and $\delta_k = \varepsilon_k$ otherwise, then $D(x) < \widehat{D}(x)$ for every $x \in (0,1)$ and $D'(x) < \widehat{D}'(x)$ for every $x \in \left(0, \frac{n}{n+1}\right)$.*

Proof. Part (a) is obvious. To prove (b), notice that $\widehat{D}(x) - D(x) = 2x^n(1-x)$, and this is positive for every $x \in (0,1)$, while its derivative is $2(n+1)x^{n-1}\left(\frac{n}{n+1} - x\right)$, so it is positive for all $x \in \left(0, \frac{n}{n+1}\right)$. □

If f is a unimodal map, denote by a the fixed point in the decreasing lap.

Now we are ready to prove that the second positive zero of the kneading determinant of a unimodal map f, if it exists, is greater than or equal to $\sqrt[3]{1/2}$. This will follow from a series of propositions. In each of them we will consider only unimodal maps with kneading sequences of a particular form, as follows.

- The cases $K = RLR^{\infty}$ and $K = RLR^{2N}L\dots$ with $N \geq 1$, will be dealt with in Proposition 4.3.
- The case $K = RL^{\infty}$ is covered by Proposition 4.4.
- The case $K = RL^NR\dots$, where $N \geq 2$, is covered by Proposition 4.7 (in this proposition we prove even a stronger result – we find the best lower bound for the second zero as a function of N).
- The case $K = RLR^{2N-1}L\dots$ with $N \geq 1$, is covered by Proposition 4.11.

Then the main result of this section is Theorem 4.12.

Note that here we are basically using the version of the kneading theory that does not use the symbol C (see Subsection 1.2).

Proposition 4.3. *($K = RLR^{\infty}$ or $K = RLR^{2N}L\dots$, $N \geq 1$) Let f be a unimodal interval map, with the kneading sequence K equal to RLR^{∞} or beginning with*

$RLR^{2N}L\dots$ with $N \geq 1$. Then the kneading determinant D of f has exactly one zero x_0 in the interval $[0,\xi)$, where ξ is the second positive zero of the polynomial $2x^7 - x^2 - x + 1$ (approximately 0.816041, which is larger than $\sqrt[3]{1/2}$). Moreover, $D'(x_0) < 0$.

Proof. If the kneading sequence of f is RLR^∞, then $D(x) = \frac{1-2x^2}{1+x}$, so it has a unique zero in $(0,1)$, namely $\sqrt{1/2}$ (which lies in $[0,\xi)$). Otherwise, since K begins with $RLR^{2N}L\dots$ and every kneading sequence is maximal, it follows that K cannot have subsequences of two consecutive Ls. In particular, K begins with $RLR^{2N}LR$.

Therefore, the sequence of signs of the power series for the kneading determinant D of f (including a $+$ for the leading 1) begins with $+--(+-)^N-+$ and it has no blocks of the same sign of length larger than 2.

We will modify this sequence of signs by induction, using Lemma 4.2(b), to obtain a power series D^+, corresponding to the sequence of signs $+--(+-)^N(-++)^\infty$, that verifies the assumptions of Lemma 4.1 for appropriate α, β and γ. This means that

$$\begin{aligned} D^+(x) &= 1 - x - x^2 + x^3(1 - x + \ldots + x^{2N-2} - x^{2N-1}) \\ &\quad + x^{2N+3}(-1 + x + x^2 - x^3 + x^4 + x^5 - \ldots) \\ &= 1 - x - x^2 + x^3\frac{1 - x^{2N}}{1+x} + x^{2N+3}\frac{-1 + x + x^2}{1 - x^3} \\ &= \frac{1-2x^2}{1+x} - 2x^{2N+3}\frac{1 - x^2 - x^3}{(1+x)(1-x^3)}. \end{aligned}$$

Clearly, $D, D^+ \in \mathscr{D}$, and by Lemma 4.2,

$$D(x) \leq D^+(x) \text{ for } x \in (0,1), \text{ and } D'(x) \leq (D^+)'(x) \text{ for } x \in (0,7/8) \tag{7}$$

(the minimum power of x affected by the rule $(-+) \mapsto (+-)$ is 7 or higher).

To simplify computations, we are going to use Lemma 4.1 not for D and D^+, but for

$$\overline{D}(x) := (1+x)D(x) \quad \text{and} \quad \overline{D^+}(x) := (1+x)D^+(x),$$

for $x \in [0,1)$. Since $D, D^+ \in \mathscr{D}$, also $\overline{D}, \overline{D^+} \in \mathscr{D}$. Moreover, from (7),

$$\overline{D}(x) \leq \overline{D^+}(x) \text{ for } x \in (0,1) \text{ and } \overline{D}'(x) \leq (\overline{D^+})'(x) \text{ for } x \in (0,7/8) \tag{8}$$

because, for $x \in (0,7/8)$,

$$(\overline{D})'(x) = D(x) + (1+x)D'(x) \leq D^+(x) + (1+x)(D^+)'(x) = (\overline{D^+})'(x).$$

Since

$$\overline{D^+}(x) = (1+x)D^+(x) = 1 - 2x^2 - 2x^{2N+3}\frac{1 - x^2 - x^3}{1 - x^3},$$

by differentiating we get

$$(\overline{D^+})'(x) = -4x - (4N+6)x^{2N+2}\frac{1 - x^2 - x^3}{1 - x^3} + 2x^{2N+3}\frac{2x + x^4}{(1-x^3)^2}.$$

Let γ be the positive zero of the polynomial $1 - x^2 - x^3$ (approximately, 0.754878). This polynomial is decreasing for $x > 0$, so it stays positive for $x \in (0,\gamma)$. Therefore, for $x \in (0, \gamma)$,

$$(\overline{D^+})'(x) < -4x + 2x^{2N+3} \frac{2x + x^4}{(1-x^3)^2} \le -4x + 2x^5 \frac{2x + x^4}{(1-x^3)^2} = \frac{P(x)}{2x(1-x^3)^2},$$

with $P(x) = x^8 - 2x^6 + 2x^5 + 4x^3 - 2$.

We want to show that $P(x) < 0$ for $x \in (0, \gamma)$. We have

$$P'(x) = 8x^7 + 10x^4 + 12(x^2 - x^5) > 0$$

for $x \in (0, 1)$, so it remains to show that $P(\gamma) < 0$. However,

$$P(x) = (x^3 + x^2 - 1)(x^5 - x^4 - x^3 + 4x^2 - 5x + 8) - (4x - 3)(x + 2),$$

and $4\gamma - 3 > 0$, so indeed, $P(\gamma) < 0$, and consequently, $(\overline{D^+})'(x) < 0$ for $x \in (0, \gamma)$.

Since $\gamma < 7/8$, by (8) we have

$$\overline{D}(x) \le \overline{D^+}(x) \text{ for } x \in (0,1) \text{ and } \overline{D}'(x) \le (\overline{D^+})'(x) < 0 \quad \text{for } x \in (0, \gamma). \tag{9}$$

Thus, in order to use Lemma 4.1 for $\overline{D}$ and $\overline{D^+}$, we need to define α and β such that $0 < \alpha < \gamma < \beta < 1$ and $\overline{D^+}(x) < 0$ for $x \in (\alpha, \beta)$. It is easy to check that $\overline{D^+}(\gamma) < 0$, so $\overline{D^+}$ has exactly one zero in $(0, \gamma)$. Call this zero α. The limit of $\overline{D^+}(x)$ as x increases to 1 is plus infinity, so $\overline{D^+}$ has the second largest positive zero, $\beta \in (\gamma, 1)$. With those definitions of α and β, our assumptions are satisfied. Thus, the first positive zero x_0 of $\overline{D}$ (and therefore, of D) exists and is in $(0, \alpha]$, and the second positive zero of $\overline{D}$ (and therefore, of D), if it exists, is larger than or equal to β. Moreover, by Lemma 4.1 applied to $\overline{D}$ and $\overline{D^+}$, $(\overline{D})'(x_0) < 0$. Then also $D'(x_0) < 0$ as required (this is obvious, since $\overline{D}(x_0) = 0$ gives $D(x_0) = 0$ and we have $(\overline{D})'(x) = D(x) + (1 + x)D'(x)$).

Here α and β depend on N. However, if $x \in (\gamma, 1)$, then $\overline{D^+}(x)$ decreases as N increases, so $\beta \ge \widehat{\beta}$, where $\widehat{\beta}$ is the value of β obtained for $N = 1$. For $N = 1$ we have

$$\begin{aligned}\overline{D^+}(x) = 1 - 2x^2 - 2x^5 \frac{1 - x^2 - x^3}{1 - x^3} &= \frac{2x^8 + 2x^7 - x^3 - 2x^2 + 1}{1 - x^3} \\ &= (2x^7 - x^2 - x + 1)\frac{1 + x}{1 - x^3},\end{aligned}$$

so $\widehat{\beta} = \xi$. This completes the proof. □

Proposition 4.4. *($K = RL^\infty$) Let f be a unimodal interval map with the kneading sequence RL^∞. Then the kneading determinant of f has exactly one zero in $[0, 1)$, namely $1/2$, and its derivative at this zero is negative.*

Proof. The kneading determinant is $1 - x - x^2 - x^3 - \cdots = (1 - 2x)/(1 - x)$. The result follows. □

The next case to consider is a kneading sequence of the form $K = RL^N R \ldots$ with a finite $N \ge 2$. To make the proof of Proposition 4.7 more transparent, we first prove the following lemmas.

Lemma 4.5. *Let $N \ge 2$ and let f be a unimodal map such that the kneading sequence of f begins with $RL^N(RL^{N-1}R)^n RL^N$ for some $n \ge 0$. Then the second zero of the kneading determinant of f, if it exists, is strictly larger than $\sqrt[N+1]{1/2}$.*

Proof. Since the kneading sequence K begins with $RL^N(RL^{N-1}R)^nRL^N$, it is straightforward to check that, regardless of whether $n=0$ or $n\geq 1$,

$$RL^{N-1}*L^\infty=(RL^{N-1}R)^\infty<K<RL^N(RL^{N-1}R)^\infty=RL^{N-1}*RL^\infty.$$

Thus, by [CE, Theorem II.2.7 (i)], there is a unique kneading sequence B such that $K=RL^{N-1}*B$ (due to the inequality $K<RL^{N-1}*RL^\infty$ we have $B\neq RL^\infty$; we will use this later). Therefore, since $D_{RL^{N-1}}(x)=(1-x-x^2-\cdots-x^N)/(1-x^{N+1})$, according to formula (5) we have

$$D_K(x)=(1-x^{N+1})D_{RL^{N-1}}(x)D_B(x^{N+1})=(1-x-x^2-\cdots-x^N)D_B(x^{N+1}). \tag{10}$$

The polynomial $P(x)=1-x-x^2-\cdots-x^N$ is decreasing for $x>0$, $P(0)=1$, and $P(1)<0$, so it has a unique zero z in $(0,1)$. If $x=\sqrt[N+1]{1/2}$, then

$$P(x)=1-x\frac{1-x^N}{1-x}=1-x\frac{1-\frac{1}{2x}}{1-x}=1-\frac{x-\frac{1}{2}}{1-x}=\frac{\frac{3}{2}-2x}{1-x}.$$

Since $N+1\geq 3$, we have $\sqrt[N+1]{1/2}>3/4$, so $P(x)<0$. Thus, $z<\sqrt[N+1]{1/2}$. On the other hand, by Theorem 2.4, the function H is strictly increasing at RL^∞ and since $B\neq RL^\infty$, the entropy of a unimodal map with the kneading sequence B is strictly smaller than $\log 2$. Hence, by the fact (F1) in Subsection 1.3, the first zero of $D_B(x)$, if it exists, is strictly larger than $1/2$. Thus, the first zero of $D_B(x^{N+1})$, if it exists, is strictly larger than $\sqrt[N+1]{1/2}$ (which in turn is strictly larger than z). Thus the second zero of D_K, if it exists, is strictly larger than $\sqrt[N+1]{1/2}$. □

Lemma 4.6. *Let $N\geq 2$ and let f be a unimodal map such that the kneading sequence of f begins with RL^NR, but not with RL^NRL^N. Then the derivative of the kneading determinant of f is negative in $(0,d)$, where d is the unique zero in $(0,1)$ of the polynomial*

$$Q(x)=-12x^7+24x^6-12x^5-6x^4+8x^3-1$$

(approximately 0.663744*).*

Proof. If $N=2$, then the sequence of signs corresponding to the kneading determinant D of f begins with $+(-)^3(+)^k-$, where $1\leq k\leq 2$. By replacing minuses by pluses and $-+$ by $+-$ (as we did in the proof of Proposition 4.3), we can get the sequence of signs $+---++-(+)^\infty$. The corresponding power series is

$$\widehat{D}(x)=1-x-x^2-x^3+x^4+x^5-x^6+\sum_{i=7}^\infty x^i=\frac{1-2x+2x^4-2x^6+2x^7}{1-x}.$$

By Lemma 4.2 we get $D'(x)\leq\widehat{D}'(x)$ for all $x\in(0,5/6)$.

If $N>2$, then the sequence of signs corresponding to the kneading determinant D of f begins with $+(-)^4$, so again by replacing minuses by pluses and $-+$ by $+-$ we can get the same sequence $+---++-(+)^\infty$. Therefore, again we get $D'(x)\leq\widehat{D}'(x)$, this time for all $x\in(0,4/5)$.

We have $\widehat{D}'(x)=Q(x)/(1-x)^2$, and Q has unique zero d in $(0,1)$. Moreover, $Q(x)<0$ for all $x\in(0,d)$. Since $d<4/5$, we get $D'(x)<0$ for all $x\in(0,d)$. □

Proposition 4.7. *($K=RL^NR\dots,N\geq 2$) Let $N\geq 2$ and let f be a unimodal map such that the kneading sequence of f begins with RL^NR. Then the second zero of* *the kneading determinant of f, if it exists, is larger than or equal to $\sqrt[N+1]{1/2}$. The equality holds if and only if the kneading sequence is $RL^N(RL^{N-1}R)^\infty$. Moreover, the derivative of the kneading determinant of f at the first zero is negative.*

Proof. Denote by K the kneading sequence of f and by D its kneading determinant. As we noticed in the introduction, in the sequence of signs corresponding to K there cannot be more than $N+1$ of the same signs in a row.

First consider the important particular case when the kneading sequence is

$$K^* = RL^N(RL^{N-1}R)^\infty = RL^{N-1} * RL^\infty.$$

Then, according to (10), the kneading determinant is

$$\begin{aligned} D^*(x) &= (1 - x - x^2 - \ldots - x^N)D_{RL^\infty}(x^{N+1}) \\ &= \frac{(1 - x - x^2 - \ldots - x^N)(1 - 2x^{N+1})}{1 - x^{N+1}}. \end{aligned} \tag{11}$$

Denote by α^* the unique positive zero of the polynomial $1 - x - x^2 - \ldots - x^N$ and set $\beta^* = \sqrt[N+1]{1/2}$. If d is the number from Lemma 4.6, then

$$\alpha^* \le \frac{\sqrt{5}-1}{2} < d < \sqrt[3]{1/2} \le \beta^* \tag{12}$$

because $(\sqrt{5}-1)/2$ is the positive zero of the polynomial $1 - x - x^2$. Observe that α^* is the first zero and β^* is the second zero of D^*, and D^* is negative between them (see Figure 7). By Lemma 4.6, the derivative of D^* at the first zero is negative. This

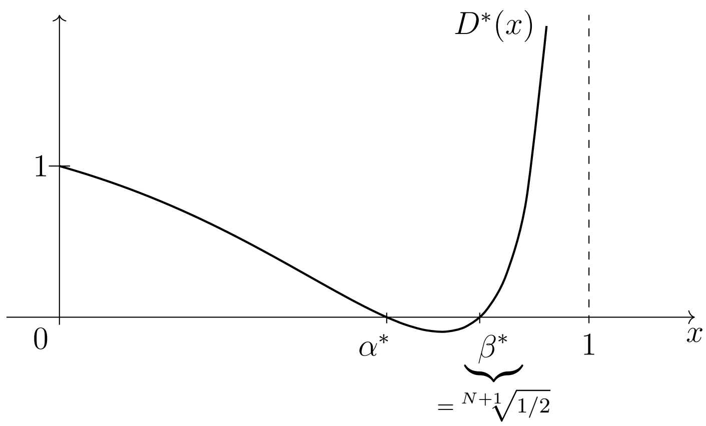


FIGURE 7. If $K^* = RL^N(RL^{N-1}R)^\infty$ then the 2nd zero of the kneading determinant $D^*(x)$ is $\beta^* = \sqrt[N+1]{1/2}$, so this lower bound for the second zero is attainable.

completes the proof in this particular case.

Now, if $K = RL^NR\ldots$ is different from the considered particular case, then there is a finite $n \ge 0$ such that

$$K = RL^N(RL^{N-1}R)^n A_1 A_2 \ldots A_{N+1} \ldots$$

where $A_1A_2\dots A_{N+1} \neq RL^{N-1}R$. We are going to prove that in this case the second zero of the kneading determinant, if it exists, is strictly larger than $\beta^* = \sqrt[N+1]{1/2}$. By Lemma 4.5, this is true if $A_1A_2\dots A_{N+1} = RL^N$. From now on we will therefore additionally assume that $A_1A_2\dots A_{N+1} \neq RL^N$, so $A_1A_2\dots A_N \neq RL^{N-1}$.

The corresponding sequence of signs is

$$+(-)^{N+1}\big((+)^N-\big)^n a_1a_2\dots a_N\dots, \tag{13}$$

and the block $a_1a_2\dots a_N$ is different from $(+)^N$. It follows that this block begins with at most $N-1$ pluses (moreover, we know that $a_1 = -$ is possible only if $n \geq 1$, since for $n = 0$ we would have $N+2$ minuses in a row). By using the rule $(-+) \mapsto (+-)$ and taking ino account that there are no blocks of pluses longer than $N+1$, by using an argument very similar to that in the proof of Proposition 4.3, we always can get the sequence

$$+(-)^{N+1}\big((+)^N-\big)^n\big((+)^{N-1}-++\big)^\infty. \tag{14}$$

By Lemma 4.2, for the kneading determinant $D(x)$ corresponding to (13) and its majorant $D^+(x)$ corresponding to (14), we have

$$D(x) \leq D^+(x) \tag{15}$$

for all $x \in [0,1)$.

We have

$$\begin{aligned} D^+(x) =& 1 - x - \dots - x^{N+1} + \frac{x^{N+2}(1+x+\dots+x^{N-1}-x^N)(1-x^{(N+1)n})}{1-x^{N+1}} \\ &+ \frac{x^{(N+1)n+N+2}(1+x+\dots+x^{N-2}-x^{N-1}+x^N+x^{N+1})}{1-x^{N+2}}. \end{aligned}$$

By grouping the terms not containing n and those containing n we get

$$\begin{aligned} D^+(x) &= \frac{(1-x-\dots-x^N)(1-2x^{N+1})}{1-x^{N+1}} - \frac{2x^{(N+1)n+2N+1}(1-x)(1-x^{N+1}-x^{N+2})}{(1-x^{N+1})(1-x^{N+2})} \\ &= D^*(x) - \frac{2x^{(N+1)n+2N+1}(1-x)(1-x^{N+1}-x^{N+2})}{(1-x^{N+1})(1-x^{N+2})}. \end{aligned}$$

The polynomial $S(x) = 1 - x^{N+1} - x^{N+2}$ is decreasing for $x > 0$, with $S(0) = 1$ and $S(1) = -1$, so it has a unique zero in $(0,1)$. We have

$$S(\beta^*) = S\left(\sqrt[N+1]{1/2}\right) = 1 - 1/2 - \left(\sqrt[N+1]{1/2}\right)^{N+2} > 0.$$

Thus $S(x)$ and, consequently, also the second term, which is subtracted from $D^*(x)$, are positive for all $x \in (0,\beta^*]$. From this and (15) it follows that

$$D(x) \leq D^+(x) < D^*(x) \tag{16}$$

for all $x \in (0,\beta^*]$.

Moreover, by Lemma 4.6, $D(x)$ has negative derivative in $(0,d)$ where, by (12), $\alpha^* < d < \beta^*$. Therefore, applying Lemma 4.1 to functions D and D^*, we get that the first zero of D is in $(0,\alpha^*]$ and the second zero, if it exists, is in $[\beta^*,1)$. Hence, the derivative of D at the first zero is negative and the second zero, if it exists, is in fact strictly larger than β^*, because (16) gives $D(\beta^*) < D^*(\beta^*) = 0$. This completes the proof. $\square$

Remark 4.8. Fix $N \geq 3$. By Proposition 4.7 and the beginning of its proof, $\sqrt[N]{1/2}$ is the second zero of the kneading determinant corresponding to the kneading sequence $K^* = RL^{N-2} * RL^\infty$. Since this K^* is a star product, it is not a kneading sequence of any tent map (as mentioned in Subsection 1.4, mixing tent maps are not renormalizable). However, $RL^{N-2}C$, which is the same as $(RL^{N-2}R)^\infty$, is the kneading sequence of the tent map whose turning point is periodic with period N; denote its slope by s_N. Since this kneading sequence is R-even, the corresponding kneading determinant $(1 - x - x^2 - \cdots - x^{N-1})/(1 - x^N)$ has only one zero in $(0, 1)$. By (F1), s_N is the reciprocal of this zero. Hence, s_N is the positive zero of the polynomial $x^{N-1} - x^{N-2} - \cdots - x - 1$ (alternatively: the largest positive zero of the polynomial $(x^{N-1} - x^{N-2} - \cdots - x - 1)(x - 1) = x^N - 2x^{N-1} + 1$). Here are some approximate values: $s_3 = (1 + \sqrt{5})/2 \approx 1.61803$, $s_4 \approx 1.83929$, $s_5 \approx 1.92756$, $s_6 \approx 1.96595$.

Unlike the point $(s_N, 1/s_N)$ on the curve of first zeros, the point $(s_N, \sqrt[N]{1/2})$, see Figure 8, does not belong to the Real Teapot (there is no second zero in this case). However, it belongs to its closure, since the limit of the second zero at the point s_N from the right (in the family of tent maps) is $\sqrt[N]{1/2}$ (see Section 5).

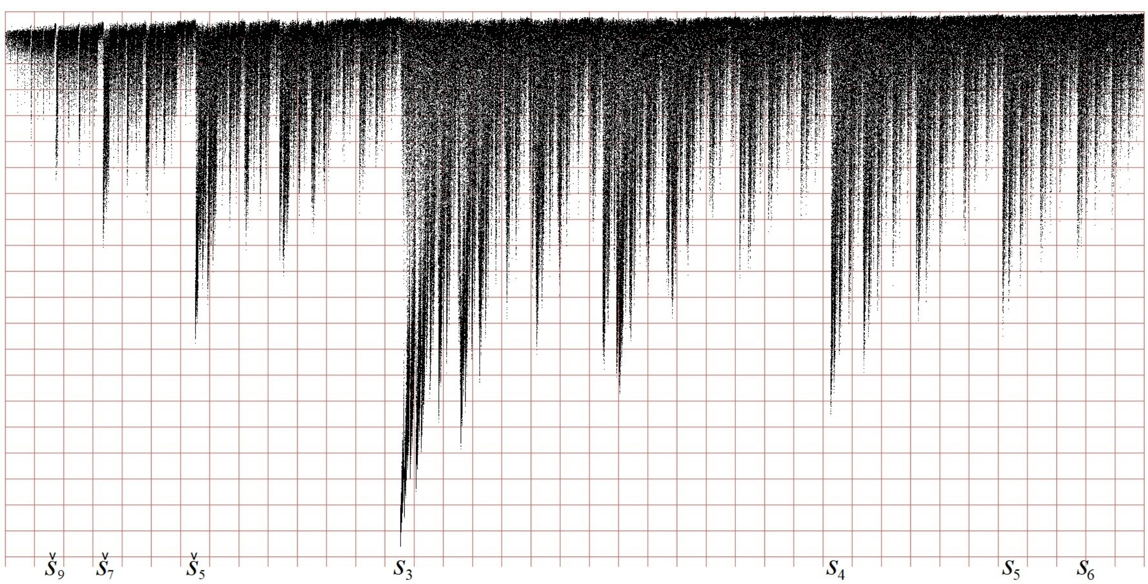


FIGURE 8. All zeros except the first ones, for the slopes $s \in [\sqrt{2}, 2]$ and with s_3, s_4, s_5, s_6 marked. We also marked slopes $\check{s}_9, \check{s}_7, \check{s}_5$, that correspond to the orbit of the turning point being Štefan of period $9, 7, 5$ (see Subsection 7.1).

From Corollary 2.5 and Propositions 4.3, 4.4, and 4.7, we get immediately the following corollary.

Corollary 4.9. *If a unimodal map has topological entropy from* $[(1/2)\log 2, \log 2]$, *then the second zero of its kneading determinant, if it exists, is larger than or equal to* $\sqrt[3]{1/2}$.

It remains to consider the case of a unimodal map g such that $K_g = RLR^{2N-1}L\ldots$ with finite $N \geq 1$.

Remember that by the zeros of a kneading determinant we mean, as always in this paper, the zeros in $(0,1)$. Unimodal maps are assumed to be defined on the interval $[0,1]$ which is their core.

Lemma 4.10. *Let $g\colon [0,1] \to [0,1]$ be a unimodal map with topological entropy $h(g) \in (0,(1/2)\log 2)$. Then there is a subinterval J of $[0,1]$ with $g^2(J) \subset J$, and a unimodal map $g^*\colon [0,1] \to [0,1]$, conjugate to $g^2|J$, such that*

(a) *$h(g^*) = 2h(g)$,*
(b) *the second zero z_2^* of D_{g^*} exists if and only if the second zero z_2 of D_g exists, and in such a case $z_2^* = z_2^2$,*
(c) *if z_1^*, z_1 are the first zeros of D_{g^*}, D_g respectively, and $(D_{g^*})'(z_1^*) < 0$ then $(D_g)'(z_1) < 0$.*

Proof. By Corollary 2.5 we have $g(0) > a$, where a is the fixed point of g in the decreasing lap. Then g is 2-renormalizable with restrictive intervals $J_0 = [0,a]$ and $J_1 = [a,1]$. We have $g(J_0) = J_1$, $g(J_1) = J_0$ and the maps $g_0 = g^2|J_0$ and $g_1 = g^2|J_1$ are topologically conjugate through the conjugacy $g|J_1\colon J_1 \to J_0$. It follows that $h(g_0) = h(g_1) = 2h(g)$.

The map g_1, after restricting it to its core $J \subset J_1$, is unimodal in our sense, so it is conjugate to a unimodal map $g^*\colon [0,1] \to [0,1]$. Then $h(g^*) = h(g_1) = 2h(g)$ and we get (a).

By (6),

$$D_g(x) = (1-x)D_{g^*}(x^2), \tag{17}$$

which tells us immediately that (b) is true.

By formula (17), we have $z_1^* = z_1^2$, and

$$(D_g)'(z_1) = -D_{g^*}(z_1^*) + 2z_1(1-z_1)(D_{g^*})'(z_1^*) = 2z_1(1-z_1)(D_{g^*})'(z_1^*) < 0,$$

and (c) follows. □

Proposition 4.11. *($K = RLR^{2N-1}L\ldots$, $N \geq 1$) Let f be a unimodal map with positive topological entropy and kneading sequence $K_f = RLR^{2N-1}L\ldots$, $N \geq 1$. Then there is $k \geq 1$ such that $h(f) \in \left(\frac{1}{2^{k+1}}\log 2, \frac{1}{2^k}\log 2\right]$. If the kneading determinant D_f has at least two zeros in $(0,1)$ then its second zero is greater than or equal to $\left(\sqrt[3]{1/2}\right)^{\frac{1}{2^k}}$ (and hence strictly larger than $\sqrt[3]{1/2}$). The derivative of D_f at the first positive zero is negative.*

Proof. By Corollary 2.5, $h(f) \in (0,(1/2)\log 2)$ and $f(0) > a$, where a is the fixed point of f in the decreasing lap. So there exists $k \geq 1$ such that $h(f) \in \left(\frac{1}{2^{k+1}}\log 2, \frac{1}{2^k}\log 2\right]$ (when $k=1$ then even $h(f) \in \left(\frac{1}{2^{k+1}}\log 2, \frac{1}{2^k}\log 2\right)$). Now we use Lemma 4.10 inductively, and at the last step (when the entropy is larger than or equal to $(1/2)\log 2$) Corollary 4.9. □

Propositions 4.3, 4.4, 4.7 and 4.11 immediately give us the following easy to remember fact.

Theorem 4.12. *Let f be a unimodal interval map with positive topological entropy. Then the derivative of the kneading determinant at the first positive zero is negative, so it is a simple zero. The second positive zero of the kneading determinant of f,* *if it exists, is greater than or equal to* $\sqrt[3]{1/2}$. *It is equal to* $\sqrt[3]{1/2}$ *if and only if the kneading sequence of* f *is* $RLL(RLR)^\infty$.

Lemma 4.10 shows that the Real Teapot has the block structure as expressed in the following corollary.

Corollary 4.13. *The Real Teapot* (1) *is a union*

$$B_0 \cup B_1 \cup B_2 \cup \dots,$$

where B_k, $k = 0, 1, \dots$ *is the set of those points of the Real Teapot whose first coordinate is in* $\left(\sqrt[2^{k+1}]{2}, \sqrt[2^k]{2}\right]$. *Moreover,*

$$\Phi(B_k) = B_{k+1} \quad \textit{and} \quad \Phi^{-1}(B_{k+1}) = B_k, \qquad k = 0, 1, \dots,$$

where $\Phi(s, x) = (\sqrt{s}, \sqrt{x})$ *and its inverse* $\Phi^{-1}(s, x) = (s^2, x^2)$.

Proof. The Real Teapot contains points $(2, 1/2)$, $\left(\sqrt{2}, 1/\sqrt{2}\right)$, $\left(\sqrt[4]{2}, 1/\sqrt[4]{2}\right), \dots$ and no other points whose first coordinate is of the form $\sqrt[2^k]{2}$, $k = 0, 1, \dots$. Indeed, for $s = 2$ and $s = \sqrt{2}$ a direct computation shows that the kneading determinant has only one zero and we know by (F1) that it is the reciprocal of the slope. For the rest of those points this follows from Lemma 4.10(b). The map Φ sends each of them to the next one. The rest follows from Lemma 4.10(b). □

To finish this section, for $s \in (1, 2]$, let us define $\varphi(s)$ as the second zero of the kneading determinant of T_s if it exists, and 1 otherwise. For all $s = \sqrt[2^k]{2}$, $k \geq 0$, the kneading determinant has only one zero in $(0, 1)$, so $\varphi\left(\sqrt[2^k]{2}\right) = 1$. The next theorem shows that the function φ is continuous at those points.

Theorem 4.14. *We have*

$$\lim_{s \to 2^-} \varphi(s) = \lim_{s \to \sqrt[2^k]{2}} \varphi(s) = 1 \tag{18}$$

for all $k \geq 1$.

Proof. As s goes to 2 from the left, the kneading sequence of T_s starts with $RL^N R$, and N goes to infinity. By Proposition 4.7, $\varphi(s) \geq \sqrt[N+1]{1/2}$, so $\varphi(s)$ goes to 1.

As s goes to $\sqrt{2}$, the kneading sequence of T_s starts with RLR^{2N} (regardless of whether the next symbol is R or L), and N goes to infinity. The corresponding sequence of signs starts with $+--(+-)^N$, so the kneading sequence is

$$D(x) = 1 - x - x^2 + (x^3 - x^4) + (x^5 - x^6) + \dots + (x^{2N+1} - x^{2N+2}) + \sum_{k=2N+3}^{\infty} \varepsilon_k x^k,$$

where $\varepsilon_k \in \{-1, +1\}$. Set

$$\begin{aligned} D^+(x) =& 1 - x - x^2 + (x^3 - x^4) + (x^5 - x^6) + \dots + (x^{2N+1} - x^{2N+2}) + \sum_{k=2N+3}^{\infty} x^k \\ =& 1 - x - x^2 + \frac{x^3 - x^{2N+3}}{1+x} + \frac{x^{2N+3}}{1-x} \\ =& \frac{1 - 2x^2}{1+x} + \frac{2x^{2N+4}}{1-x^2}. \end{aligned}$$

Then $D(x) \le D^+(x)$ and $D'(x) \le (D^+)'(x)$ for all $x \in (0,1)$.

We have

$$(D^+)'(x) = \frac{-2x^2-4x-1}{(1+x)^2} + x^{2N+3}\frac{4N+8-(4N+4)x^2}{(1-x^2)^2} \le -1 + \frac{(4N+8)x^{2N}}{(1-x^2)^2}$$

for all $x \in (0,1)$. The function $\psi_N(x) = -1 + (4N+8)x^{2N}/(1-x^2)^2$ is increasing on $(0,1)$. Fix $\varepsilon \in (0,1)$. If N is sufficiently large, then $\psi_N(1-\varepsilon) < 0$, so $D'(x) \le (D^+)'(x) \le \psi_N(x) < 0$ for all $x \in (0,1-\varepsilon)$. Therefore D has at most one zero (namely, the first one) in $(0,1-\varepsilon)$. Consequently, if s is sufficiently close to $\sqrt{2}$, then $\varphi(s) \ge 1-\varepsilon$. This proves (18) for $k=1$.

Now (18) for $k>1$ follows by Corollary 4.13. □

5. Close to period 3

Let us look at what happens close to the slope where the turning point is periodic of period 3 (this slope is the golden mean; we will denote it by s_3). Let us start by looking at Figures 9, 10 and 11. On the horizontal axis there are slopes s of the tent maps, on the vertical axis are the zeros of the kneading determinant (1 is at the top). For x close to 1 the power series that defines the kneading determinant $D_s(x)$ converges very slowly, so numerical methods for finding the zeros fail. Therefore we see a gap close to $x=1$, which is an artifact. In fact, some coarse estimates suggest that we cannot believe anything with $x > 0.99$ (the highest red squares).

In our pictures every horizontal pixel (there are 1800 of them) corresponds to 1000 values of s. The red grid makes it simpler to see approximate values of s and the zeros of D_s.

In Figure 9 the slopes are from 1.61 to 1.67. Vertical red lines are every 0.0015, and horizontal ones every 0.01 (so the highest horizontal line is at $x=1$ and the lowest one at $x=0.79$). The position of s_3 is easily visible not far from the left side of the picture. To the left of s_3 all zeros are close to 1, while to the right of it (and close to it) it looks like there are additional zeros quite far from 1. A natural conjecture is that those are second zeros.

Now let us stretch the horizontal axis. In Figure 10 the slopes are from 1.6175 to 1.6235, so the vertical red lines are every 0.00015. We see better the phenomena that we noticed in Figure 9.

Let us continue and stretch the horizontal axis even more, and plot only second (black) and third (cyan) zeros. In Figure 11 the slopes are from 1.6178 to 1.6184, so the vertical red lines are every 0.000015. And indeed, it looks like to the right of s_3 the zeros far from $x=1$ are second zeros.

However, we see clearly another phenomenon (which we could have noticed earlier). Namely, there is a change in the form of discontinuities in the zeros close to 1. In fact, we cannot tell whether they are discontinuities; what we see may be just very steep curves (however, let us refer to them as discontinuities). To the left of s_3 (where they are second zeros) they look like discontinuities from the left, while to the right of s_3 (where they are third zeros) they look like discontinuities from the right. Moreover, they correspond to the steep parts of graph for the second zeros, which are far from 1.

Let us try to explain the phenomenon that we observed. What happens immediately to the left and right of s_3? At s_3 the kneading sequence is RLC, so the

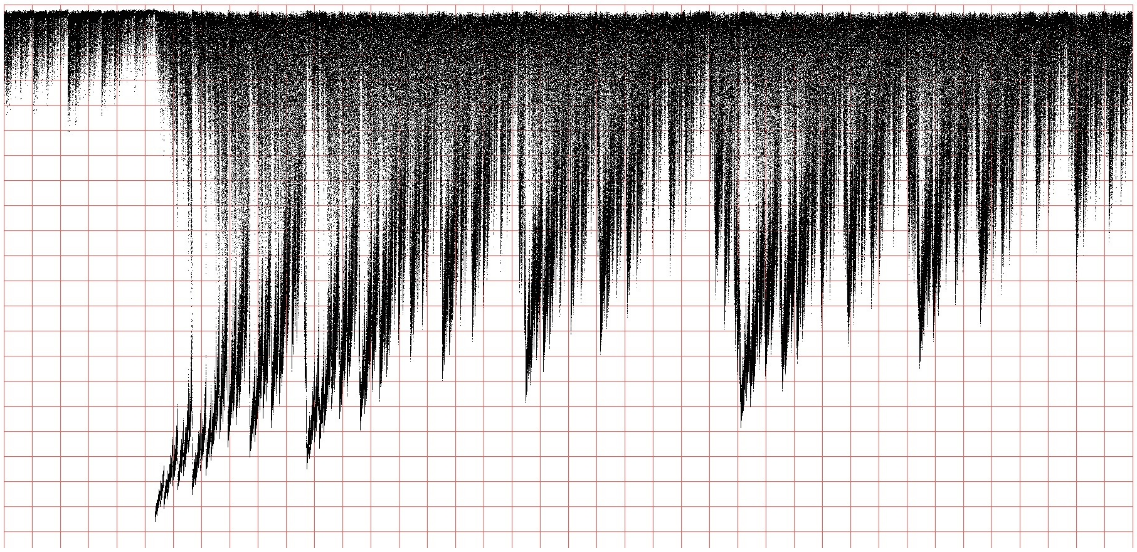


FIGURE 9. Slopes from 1.61 to 1.67, all zeros except the first ones.

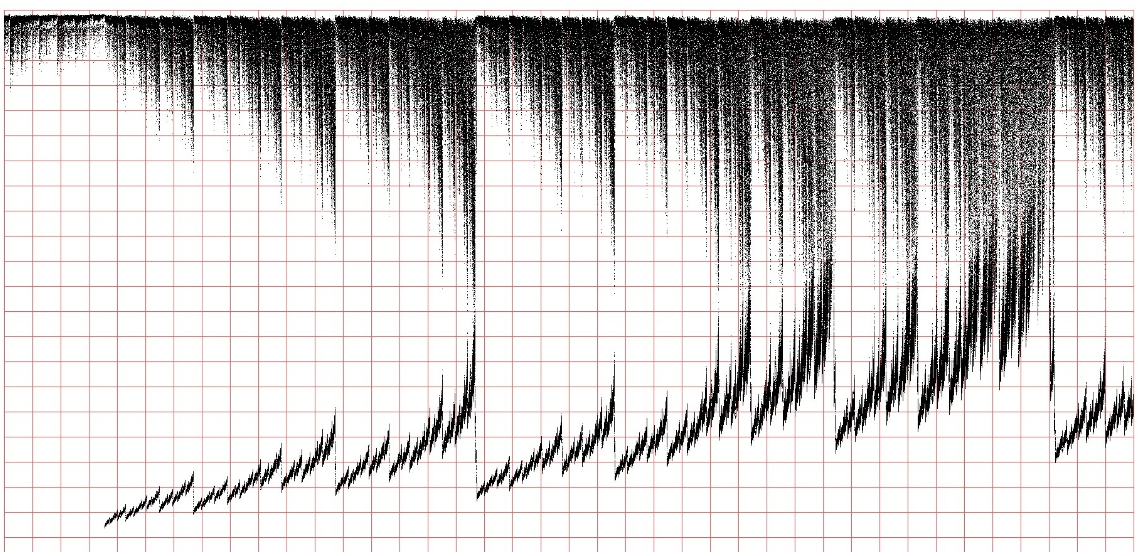


FIGURE 10. Slopes from 1.6175 to 1.6235, all zeros except the first ones.

kneading determinant is $(1-x-x^2)/(1-x^3)$. It has only one zero in $(0,1)$ (in fact, we knew it without calculations, because the period 3 orbit is twist). If s is slightly smaller, a big chunk of the power series stays the same, so for a small $\varepsilon > 0$ the kneading determinant has only one zero in $(0, 1-\varepsilon)$. If s is slightly larger than s_3, then the kneading determinant is close not to D_{s_3}, but to the kneading determinant of the map with kneading sequence which is the star product of RL and RL^∞ (the kneading sequences in between, including the last one, are missing in our picture, because there are no such tent maps). The kneading determinant of this map is $(1-x-x^2)(1-2x^3)/(1-x^3)$, so it has two zeros in $(0,1)$. The second zero is $\sqrt[3]{1/2}$.

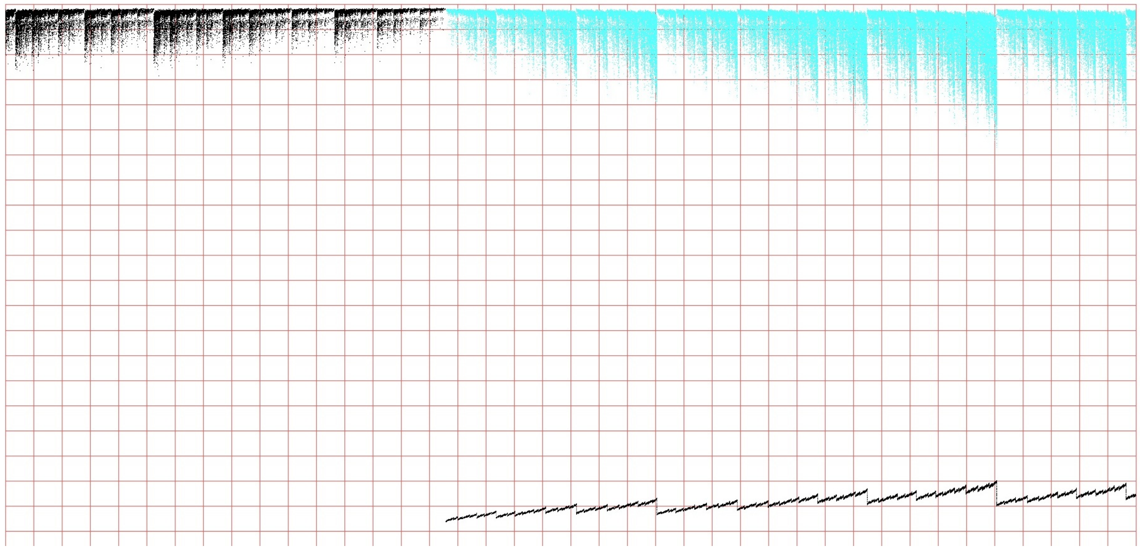


FIGURE 11. Slopes from 1.6178 to 1.6184, second and third zeros.

This means that for s slightly larger than s_3 there is a second zero close to $\sqrt[3]{1/2}$ and for a small $\varepsilon > 0$ the kneading determinant has only two zeros in $(0, 1-\varepsilon)$.

This shows that on some interval $(s_3, \widehat{s})$ or $(s_3, \widehat{s}]$ the second zero exists, so it is a function of s. This function was already introduced in connection with Theorem 4.14 and denoted there by φ. From what was said above we know that

$$\lim_{s \to (s_3)^+} \varphi(s) = \sqrt[3]{1/2}.$$

The function φ is continuous, at least at points from $(s_3, \widehat{s})$. Indeed, the only values of s at which the kneading determinant as a function of s is discontinuous, are those at which the turning point is periodic (in the space of kneading determinants we take the topology of uniform convergence on compact subsets of $[0, 1)$). If for a given s the turning point is periodic of period n, then

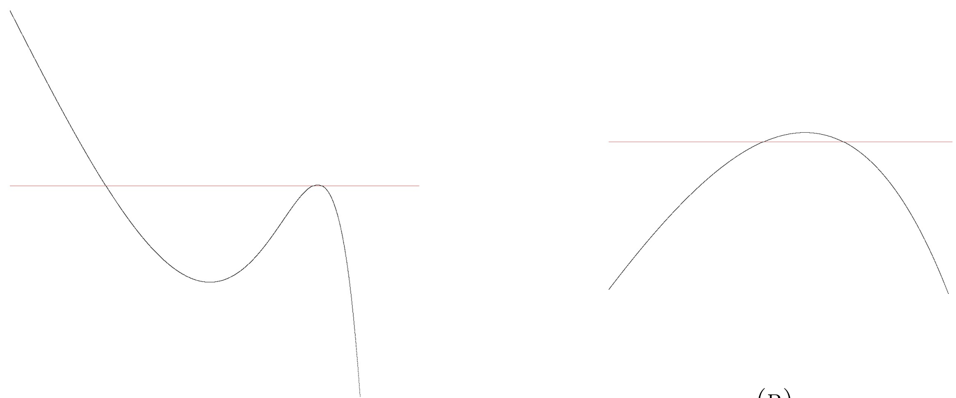
$$\lim_{t \to s^-} D_t(x) = D_s(x) \quad \text{and} \quad \lim_{t \to s^+} D_t(x) = (1 - 2x^n) D_s(x).$$

Since the zero of $1 - 2x^n$ is much larger than the second zero of D_s, the second zero of both limits is the same.

Let us take the largest $\widehat{s}$ for which the above holds. It is interesting what happens at $\widehat{s}$. There are three possibilities. The first one is that at $\widehat{s}$ the second and third zeros meet, creating a second zero of multiplicity 2 (or larger), and when s increases, this second zero disappears. Of course in this case some larger zero can become the second one. The second possibility is that in the situation described in the preceding paragraph the zero of $1 - 2x^n$ is smaller than the second zero of D_s, and it becomes the second zero. The third possibility is that the limit of the second zero at $\widehat{s}$ is 1.

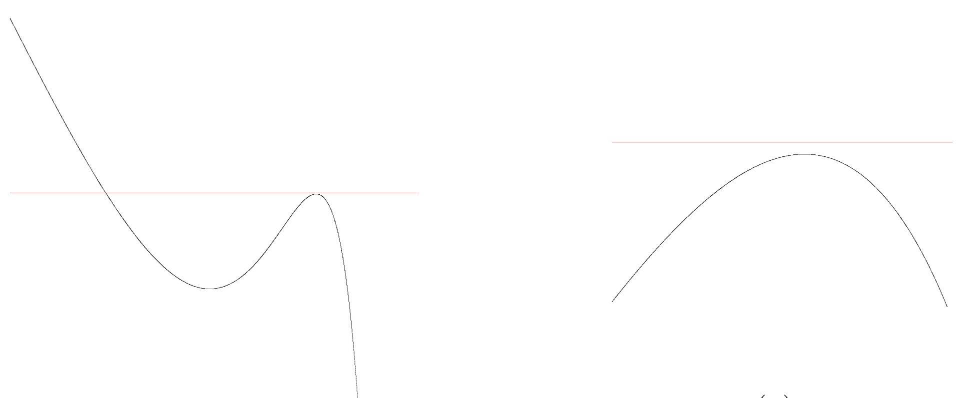
There is a strong numerical evidence that in our case the first possibility occurs. In Figures 12 and 13 we see that between 1.619961091 and 1.619961092 the second and third zeros collide.

If we start not from an s corresponding to a turning point of period 3, but from any other s for which the turning point is periodic, we have a similar situation (in particular, $\lim_{s \to (s_N)^+} \varphi(s) = \sqrt[N]{1/2}$ holds also for all $N \geq 4$), but in general the zero we consider will not be necessarily the second one. Then, other possibilities may be realized. Anyway we see that our real teapot contains the union of countably many graphs of continuous functions with various connected domains.



(A) (B)

FIGURE 12. (A) The graph of the kneading determinant for the slope 1.619961091; $0.5 \leq x < 1$. (B) The part around the turning point is zoomed in; we can see the second and the third zero.



(A) (B)

FIGURE 13. (A) The graph of the kneading determinant for the slope 1.619961092; $0.5 \leq x < 1$. (B) While the value at the turning point in Figure 12 was positive, now it is negative. Hence, for some slope between 1.619961091 and 1.619961092, the value at the turning point is equal to zero (the zero is multiple).

The pictures strongly suggest that those graphs have a nice fractal structure. It is an interesting question, what is their Hausdorff dimension (and is it the same for every curve?).

For the "discontinuities" mentioned earlier we have an explanation only if s is close to s_3 from the left. Then we have the phenomenon described above, whenever the turning point is a periodic point. However, since the period is much larger than 3, everything happens close to $x = 1$, and moreover, to see the similar picture we would have to stretch a lot the s-axis. We do not have the explanation for the "inverted discontinuities" to the right of s_3. It may be connected with the fact that these are third zeros, or again, it may only look like this because we do not stretch enough the s-axis.

6. Closure of the Real Teapot

In the definition of Thurston's Master Teapot one takes the closure. If we consider the closure of the Real Teapot (in $(1, 2] \times (0, 1)$), what would change in our results?

The only section where interesting questions arise, is Section 2. We proved there that if a unimodal map is R-even, then its kneading determinant has a unique zero in $(0, 1)$. The latter property for the map T_s is equivalent to $\{s\} \times (0, 1)$ intersecting the Real Teapot only at one point $(s, 1/s)$. If we replace the Real Teapot by its closure, this property does not necessarily hold for R-even maps. For instance, for s_3, considered in Section 5, the kneading sequence of T_{s_3} is R-even (the orbit of the turning point is even twist), but the point $\left(s_3, \sqrt[3]{1/2}\right)$ belongs to the closure of the Real Teapot.

However, such phenomenon is impossible if the orbit of the turning point of T_s is not periodic.

Theorem 6.1. *If the kneading determinant of the map T_s has only one zero and the turning point of T_s is not periodic, then $\{s\} \times (0, 1)$ intersects the closure of the Real Teapot only at one point $(s, 1/s)$.*

Proof. The function assigning to t the kneading determinant of T_t is continuous at $t = s$. Since we are dealing with analytic functions, the same is true if we replace the kneading determinant by its derivative. By Theorem 4.12, this derivative is negative at the first zero of T_s. Our topology in the space of kneading determinants is that of uniform convergence on compact sets. All this shows that for any compact subset X of $(0, 1)$ the set $\{s\} \times X$ intersects the closure of the Real Teapot at most at one point. Therefore, $\{s\} \times (0, 1)$ intersects the closure of the Real Teapot only at one point. □

Together with Theorem 2.12, we get the following corollary.

Corollary 6.2. *The set of those slopes $s \in (\sqrt{2}, 2]$ for which $\{s\} \times (0, 1)$ intersects the closure of the Real Teapot only at one point $(s, 1/s)$, has cardinality of the continuum.*

This corollary allows us to draw conclusions about the connectivity properties of the closure of the Real Teapot.

Theorem 6.3. *The closure of the Real Teapot in $(1, 2] \times (0, 1)$ has infinitely many connected components.*

Proof. By Proposition 4.11, if $s \in (2^{1/2^{k+1}}, 2^{1/2^k}]$, then the second zero of D_s, if it exists, is greater than $\left(\sqrt[3]{1/2}\right)^{1/2^k}$. Since $\left(\sqrt[3]{1/2}\right)^{1/2^k} > (1/2)^{1/2^{k+1}}$, this means that the hyperbola (strictly speaking, a part of a hyperbola) of the first zeros lies strictly below the rest of the closure of the Real Teapot. This hyperbola is one connected component. The vertical lines $\{s\} \times (0,1)$, where $s \in \mathscr{R}$, are disjoint from the rest of the closure of the Real Teapot, so they divide it into infinitely many sets. Each of those sets is either a connected component, or a union of connected components. □

Of course, since the closure of the Real Teapot has infinitely many connected components, the same is true for the Real Teapot itself.

7. Questions

7.1. **When a kneading determinat is between kneading determinants of two Štefan cycles.** We have shown in Section 4 that the second positive zero of the kneading determinant of a unimodal map with positive entropy, if it exists, is greater than or equal to $\sqrt[3]{1/2} \approx 0.793701$. In particular it is true for any kneading determinant $D_K(x)$ where

$$K_3 = (RLR)^\infty < K,$$

and K_3 is the kneading sequence corresponding to the 3-cycle. Analogously, for $k \geq 1$ and Štefan's cycle of period $2k+1$ let us denote its kneading sequence $K_{2k+1} = (RLR^{2k-1})^\infty$. By Corollary 2.10 its kneading determinant has a unique zero in $(0,1)$. We are able to show that the second positive zero of the kneading determinant $D_K(x)$ with $K_5 < K < K_3$, if it exists, is greater than or equal to $\sqrt[5]{1/2} \approx 0.870550$. Can be this observation generalized, i.e., is it true that the second positive zero of the kneading determinant $D_K(x)$ with $K_{2k+3} < K < K_{2k+1}$, if it exists, is greater than or equal to $\sqrt[2k+3]{1/2}$ (see Figure 8)?

7.2. **When the turning point is periodic.** For $n \geq 3$, the number U_n of all unimodal n-cycles (more precisely, unimodal patterns of period n, according to the terminology of [ALM]) is given by the formula

$$U_n = \frac{1}{n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{\frac{n}{d}-1},$$

where μ is the Möbius function, see [WR] or [G]. For each of them we may consider the corresponding connect-the-dots unimodal map and its kneading determinant which, since the turning point is periodic with period n, has the form

$$\frac{1 - x - x^2 + \varepsilon_3 x^3 + \cdots + \varepsilon_{n-1} x^{n-1}}{1 - x^n}$$

(with a Littlewood polynomial in the numerator, i.e. a polynomial whose all coefficients are in the set $\{-1, +1\}$). We may ask how many zeros in $(0,1)$ does the kneading determinant have. If, for given n, we check this for all U_n kneading determinants, how often do we get 1 zero, 2 zeros, etc.?

By checking periods from 3 to 23 which are prime numbers, we find that 2 zeros appear for the first time for $n = 11$ and 3 zeros for $n = 23$. Moreover, for the period 23, among $U_{23} = 182361$ cases we get 1 zero 157760 times, 2 zeros 24598 times, 3 zeros 3 times. That is, in 86.5% of cases there is only one zero. How is it for much higher periods? Further, what is the smallest period n_k when k zeros appear? If N_n is the maximal number of zeros appearing for the period n, is it true that $N_n/n \to 0$ when $n \to \infty$?

7.3. **Simple zeros.** By Theorem 4.12, for the unimodal maps, the first zero in $(0, 1)$ of the kneading determinant is always simple. In general this is not true for other zeros in $(0, 1)$, see Figure 13(B). We conjecture that for a generic slope s the kneading determinant of the tent map T_s has only simple zeros.

7.4. **Topology of the Real Teapot.** We saw in Section 5 that the Real Teapot contains the union of countably many graphs of continuous functions with various connected domains. Is it equal to such union? Does it have isolated points? If yes, how many? Is it locally connected?

Another question is about the closure of the Real Teapot in $(1, 2] \times (0, 1)$. We saw that it is not connected, but is it locally connected? And what happens if we take the closure in $[1, 2] \times [0, 1]$ instead? Is it connected? Is it locally connected?

7.5. **Galois conjugates vs. zeros of the kneading determinant.** Suppose that the turning point for T_s is periodic. Tiozzo, [T, Subsection 7.3], gives an example where not all zeros of the kneading determinant of T_s are Galois conjugates of $1/s$. However, this example is the Štefan orbit (see [ALM]) of period 7. Štefan orbits are twist, so those extra zeros do not belong to the real interval $(0, 1)$. Does there exist a similar example with the extra zeros in $(0, 1)$?

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