Dynamics on parameter spaces
Sde-Boker 2013

Open problem session

Boris Solomyak:

Let $\|\cdot\|$ denotes the distance to the closest integer.

For which $\theta > 1$ can one find $C > 0$ and $\rho = \rho(\theta) > 1$ such that for all N

$$\max_{\tau\in[1,\theta]} \exp\left(-\sum_{n=1}^{N} \|\tau\theta^n\|^2\right) < C\rho^{-N}?$$

False for

- θ Pisot (all conjugates inside the unit circle),
- θ Salem (all conjugates on unit circle except $1/\theta$),
- G_δ set.

True for almost all θ (Erdős 1940).

Theorem (Pisot). *There exists such $\tau > 0$ that $\sum_{n=1}^{\infty} \|\tau\theta^n\|^2 < \infty$ if and only if θ is Pisot number.*

Motivation: $\lambda = 1/\theta$. Let ν_λ be the distribution of the random series $\sum_{n=1}^{\infty} \pm\lambda^n$ where the signs are chosen independently with probabilities $(1/2, 1/2)$ (this is the Cantor-Lebesgue measure when $\lambda < 1/2$ and is usually called 'infinite Bernoulli convolution' for an arbitrary $\lambda < 1$). Then $\widehat{\nu_\lambda}(t) = \prod_{n=0}^{\infty} \cos(2\pi t\lambda^n)$. Answering the Question would have implications for absolute continuity and smoothness properties of ν_λ.

For more details and references see

Y. Peres, W.Schlag, B. Solomyak, Sixty years of Bernoulli convolutions. Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), 39-65, Progr. Probab., 46, Birkhäuser, Basel, 2000.

Rodrigo Treviño:

Is there a translation surface of infinite genus, which is finite with respect to the planar area form, and has a lattice Veech group?

Martin Möller:

Consider the REL foliation of a stratum $\Omega\mathcal{M}_g(\underline{\kappa})$ with leaves of complex dimension $|\underline{\kappa}| - 1$. These leaves carry a natural flat structure, and in fact, when $|\underline{\kappa}| = 2$, have the structure of a quadratic differential.

- Understand leaves with compact completion.

Theorem of M. Schmoll: Whenever a surface is square-tiled with $|\underline{\kappa}| = 2$, the leaf completion is compact and square-tiled. In $\Omega\mathcal{M}_2(1,1)$ if the leaf is over a square-tiled surface then the Veech group of the completion is $\mathrm{Sl}(2, \mathbf{Z})$. Are there other compact square-tiled leaf-completions?

- What are closures of leaves (in strata/in moduli space)?

  McMullen has an example of a straight trajectory in a leaf with *wild* closure.

  Over primitive Teichmüller curves i.e. when a leaf is contained in $\Omega\Sigma_D$ then $\mathbf{P}\Omega\Sigma_D = \mathbf{H} \times \mathbf{H} / \mathrm{Sl}(2, \mathcal{O}_D))$ and by Mautner all leaves are dense.

- Is the REL-foliation ergodic in every stratum with $|\underline{\kappa}| \geq 2$?

Barak Weiss:

Is there a classification of square-tiled surfaces with Veech group equal to $\mathrm{Sl}(2, \mathbf{Z})$?

Mike Hochman:

Let $\Pi_n = \left\{ \sum_{k=0}^{n} \sigma_k x^k | \sigma_k \in \{0, \pm 1\} \right\}$. Does there exist c such that if α and β are real roots of some polynomials from Π_n then either $\alpha = \beta$ or $|\alpha - \beta| > c^n$?

Motivation: Let ν_λ denote the distribution measure of the random series $\sum_{n=1}^{\infty} \pm 1 \lambda^n$, where the signs are chosen i.i.d. with equal probabilities (this is the Bernoulli convolution with parameter λ). The measure ν_λ has dimension 1 if $\nu(E) = 0$ for every Borel set E of Hausdorff dimension < 1. A positive answer to the question above would imply that $\dim(\nu_\lambda) = 1$ unless the parameter λ is algebraic. The best current result allows a dimension 0 (but possibly uncountable) set of such parameters.

Uri Shapira:

Let

$$\mathbf{QI} = \{\alpha \in \mathbf{R} : \alpha \text{ is a quadratic irrational}\}.$$

Given $\alpha \in \mathbf{QI}$ we denote by $a_i(\alpha)$ the i'th digit of the continued fraction expansion of α and by ν_α the normalized counting measure supported on the (eventual) period of α under the Gauss map T in the unit interval ($T : x \mapsto 1/x - \lfloor 1/x \rfloor$). Let ν denote the Gauss-Kuzmin measure on the unit interval.

Definitions. Let $\alpha_n \in \mathbf{QI}$ be a sequence.

(*i*) We say that α_n is *asymptotically Gauss-normal* if ν_{α_n} converges in the weak* topology to ν.

(*ii*) We say that α_n is *uniformly bounded* if there exists M such that $\sup_n \limsup_k a_k(\alpha_n) \leq M$.

(*iii*) We say that α_n is *uniformly divergent* if $\liminf_n \liminf_k a_k(\alpha_n) = \infty$.

Problem. Let $\alpha \in \mathbf{QI}$.

- Can one always find a sequence of primes p_n such that $p_n\alpha$ is asymptotically Gauss-normal?
- Can one always find a sequence of primes p_n such that $p_n\alpha$ is uniformly bounded?
- Can one always find a sequence of primes p_n such that $p_n\alpha$ is uniformly divergent?