Memory loss can prevent chaos in games dynamics

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We study the dynamics of simple congestion games with two resources, where a continuum of agents behaves according to a simplified version of Experience-Weighted Attraction (sEWA) algorithm. Dynamics is characterized by the population intensity of choice/learning rate $a > 0$, capturing their economic rationality, i.e., their tendency to approximately best respond to the other agent's behavior, and a discount factor (exploration parameter) $\sigma \in [0, 1]$, capturing a type of memory loss (recency bias) where past outcomes matter exponentially less than the recent ones. Finally, our system adds a third parameter $b \in (0, 1)$, which captures the asymmetry of the cost functions. We show that for any discount factor σ the system is destabilized for a sufficiently large intensity of choice a; however, dependent on the level of the asymmetry of the game b, and its relation to σ, the system stays predictable or becomes unpredictable and chaotic. As σ increases the chaotic regime gives place to a periodic orbit of period 2 that is globally attracting except for a countable set of points that lead to equilibrium. Therefore, as the discount factor increases, memory loss can make the system predictable.

Learning in games is a universal modelling tool used in game theory, economics, mathematical biology and artificial intelligence. Nevertheless, even in the simple setting of a congestion game with two resources and full memory, under minimal genericity assumption chaos is inevitable under sufficiently large learning rate[24]. But what if agents forget? We study what happens if agents discount their past, where past outcomes matter exponentially less than the recent ones. We show that as agents forget more (the recency bias grows) the chaotic regime gives place to predictable behavior of period 2, almost as predictable as the gold standard of a globally attracting equilibrium.

I. INTRODUCTION

Congestion games[65] are arguably amongst the most well-studied classes of games in game theory. They capture multi-agent settings where the costs of each agent depends on the resources she chooses and how congested each of them is (e.g., traffic routing, common resources). Congestion games are isomorphic to potential games[59], i.e., games where the incentives of all agents are perfectly aligned with each other by being equivalent to optimizing a single potential function. Furthermore, population (non-atomic) congestion games are even more regular game settings as under a minimal natural assumption on the cost functions of resources it is known that they admit an essentially unique equilibrium flow which coincides with the global minimum of a strictly convex potential function[60]. Given the above, they are typically thought as the paragon of game theoretic stability with numerous evolutionary dynamics provably converging in them via Lyapunov arguments where the potential function is strictly decreasing with time[20,26,32,33,46,47,56,57,61,68].

Despite the ubiquitous nature of these positive results, at a closer look, a common driving "regularity" assumption emerges at their core. The behavioral dynamics have to be in a sense "smooth" enough to act as a gradient-like system for the common potential function. This type of regularities typically follow automatically in the case of continuous-time dynamics[68] whereas in the case of discrete-time dynamics they can be enforced by appropriate upper bounds on the step-size/learning rates. These bounds decrease as we increase the Lipschitz constant of the gradient of the potential with steeper potentials resulting in more restrictive bounds on the intensity of choice of the agents. In the case of non-atomic congestion games, as we increase the total population size (i.e., total load/congestion) these bounds converge to zero. Of course, the mathematical necessity of such regularity conditions is abundantly clear as even with gradient descent on a strictly convex function the step-size has to be controlled as the function becomes steeper to avoid overshooting effects. What is less clear is how well do these mathematically driven constraints agree with our best known understanding of how people actually behave and adapt when facing such strategic considerations in practice:

What type of behavior emerges if we move away from assumption of vanishingly small step-sizes?

Does equilibration persist and if not what takes its

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place?

Motivated by this set of questions a number of recent papers have studied standard online learning algorithms such as Multiplicative Weights Update (MWU) (and variants thereof) in congestion games[9,24,25,61]. The main insight of the line of research is that even in the case of rather simple non-atomic congestion games, with only two strategies/resources/paths available for the agents and even with linear cost for each path, if the effective population learning rate (i.e., learning rate × population size) becomes large enough then the system becomes unstable[24]. Furthermore, under a minimal genericity assumption that the cost functions are not perfectly symmetric, i.e., the unique equilibrium flow is a perfect 50-50 split, then the system will not only become unstable but actually formally chaotic. Similar chaotic results has since been established for many other class of games and applications such as Cournot competition, Fisher markets and auctions[21] or transaction fee mechanism design[49]. This proliferation of results raises the following question:

Is chaos inevitable in congestion games under sufficiently large learning rate?

The question of how people learn to adapt their strategies in real-world games is the object of study of behavioral game theory[17,18,42]. Experience-Weighted Attraction (EWA) is a canonical learning model in behavioral game theory and although in its full generality contains too many free parameters recent work has focused on a stripped down version[36], which we will call simplified EWA (sEWA). It allows only two free parameters, the intensity of choice, a, (akin to the exponent of a logit choice function[2]) and a memory loss parameter, σ, which is akin to the rate of exponential discounting over past payoffs. Equivalently, our dynamics can be interpreted as the standard MWU algorithm where the agents' costs have been regularized by adding an appropriately scaled negative entropy term, which encourages exploration of suboptimal strategies. We adopt this model where, roughly speaking, agents perform logit best-responses to an estimate of the historical performance of each action where the effect of past payoffs/costs decays exponentially fast.

Importantly, experimental work in the area suggests that large intensities of choice are common[18], pointing out at the possibility of a conflict between standard mathematically driven assumptions and experimentally tested behavioral regularities. Similarly, the discounting rate is shown to be reliably positive; however, its actual value can vary widely from game to game[42]. Thus, our motivating questions translate to our setting as follows:

How does the interplay between intensity of choice and memory loss affect the convergence results in simple congestion games?

At what points of the parameter space do the dynamics destabilize and when they do what sort of behavior do they give rise to (limit cycles, chaos, etc.)?

Finally, how does the nature of the congestion game (e.g. symmetry of costs) affect its stability?

In a prior conference proceeding [24], we have provided some preliminary answers to these questions for the special case where the agents have no memory loss, whereas now we provide a thorough understanding of the complex phenomena emerging at different levels of memory loss.

Our results. We describe how memory loss and asymmetry of costs interacts in simple population games with two strategies when the increase in the intensity of choice results in losing stability of the system.

Main Theorem (informal). For a fixed discount factor $\sigma \in [0,1]$, where $\sigma = 0$ describes a full memory case and increasing $\sigma > 0$ means increasing memory loss (recency bias), we have the following:

- if the costs functions of the two strategies don't differ significantly, captured in our model by $b \in (\frac{1-\sigma}{2-\sigma}, \frac{1}{2-\sigma})$, then after losing stability (when the system equilibrium/fixed point is not longer attracting) the system stays predictable for any values of intensity of choice.
- If b is outside of $(\frac{1-\sigma}{2-\sigma}, \frac{1}{2-\sigma})$, then for sufficiently large values of intensity of choice the system becomes unpredictable.

Moreover, as memory loss increases the system becomes predictable for a wider range of cost functions.

This is a crucial differentiation of the long-term behavior of the system: in the first case the existence of a periodic orbit of period 2 which attracts almost all trajectories implies that although the system does not stabilize, it remains relatively predictable — no matter the initial state of the system, it will converge to a unique period 2 orbit, thus after some time, every even number of iterations of the map will place it close to its previous position. When the system becomes chaotic we land in an unpredictable regime at the antipode of this behavior with periodic orbits of different periods, positive topological entropy and complicated dynamics.

Due to a memory loss parameter, the (interior) fixed point of our learning dynamics corresponds to a unique quantal response equilibrium[54], reflecting the bounded rationality of the agents. As the intensity of choice increases the fixed point approaches the Nash equilibrium (Proposition IV.2) but at the cost of losing stability. As long as the fixed point is locally attracting then it is globally attracting . Moreover, there is a lower bound on the learning rate/intensity of choice, above which the fixed point is repelling. This threshold value is decreasing with the discount factor (Theorem IV.5). If there is memory loss (recency bias), then there exists a threshold value on the intensity of choice that if exceeded, then the game is unstable no matter what the costs are. In contrast, if there is no memory loss, for any intensity of choice one can find congestion games such that the dynamics is stable (Proposition IV.6).

Next, we study exactly how unpredictable the system will become when stability is lost. In the case of systems with memory, as long as the underlying congestion game is not very asymmetric, then under large enough intensity of choice the system will inevitably converge to an attracting periodic orbit of period two (Theorem IV.9). Thus, although the system won't converge, the behavior will be predictable. On the other hand, if the resources have significantly different costs resulting in a Nash equilibrium flow b far from the symmetric $50\% - 50\%$ split (i.e., the value b is far from 0.5), then there is a threshold value for the intensity of choice above which the dynamics are Li-Yorke chaotic and have positive topological entropy (Theorem IV.11). Finally, in Section IV D we show another way of interpreting our system — through the Multiplicative Weights Update algorithm with perturbed costs.

We complement our theoretical understanding with further simulations and numerical experiments. In Figure 1 we present bifurcation diagrams at increasing values of the discount factor σ in a specific instance of a two-resource congestion game where the Nash equilibrium flow is fixed at $b = 0.4$. Without any discounting ($\sigma = 0$), the dynamics leads to the period-doubling bifurcation route to chaos as one increases the intensity of choice a. As the discount factor increases to $\sigma = 0.25$, chaos starts at a much larger intensity of choice. Finally, as the discount factor increases to $\sigma = 0.5$, increasing the intensity of choice can at most result in the existence of a period-2 limit cycle. Thus, a larger discount factor tends to make the dynamics more predictable.

In Figure 2 we show how increasing the discount factor/memory loss σ can reduce chaotic, unpredictable dynamics into periodic, predictable ones by examining both the day-to-day behavior as well as the time evolution on the potential function of the game.

Finally in Appendix A we derive simplified EWA from the general EWA model, and in Appendix B we show connection of this model with Multiplicative Weights algorithm.

II. RELATED LITERATURE

Concepts discussed in this article arise in different contexts and were studied independently by many economists and computer scientists. The problem of introducing discounting of the past costs is an important issue both from theoretical and experimental economics perspective.

Experience-Weighted Attraction (EWA). EWA is arguably one of the most influential learning models in behavioral game theory[17–19]. Recently[62], it was shown that best reply cycles can predict non-convergence of six well-known learning algorithms in games with random payoffs where one of the algorithms considered is EWA. Other dynamics include replicator dynamics, reinforcement learning, fictitious play and k-level EWA, showing that often there exist similarities between their behavior at least in small, randomly chosen games. Closely related to the model discussed in this article is also (considered usually in the continuous time) perturbed best response dynamics[8,28,31,43,68].

Learning in games. Learning procedures can be divided into two broad categories depending on whether they evolve in continuous or discrete time: the former includes the numerous dynamics for learning and evolution (see Sandholm[68] and Hadikhanloo et al.[40] for recent surveys), whereas the latter focuses on learning algorithms (such as fictitious play and its variants) for infinitely iterated games[35].

The EWA algorithm, discussed in this paper, can be seen as a reinforcement learning algorithm where agents score their actions over time based on their observed payoffs and then they choose an approximate/perturbed best response. Learning algorithms of this kind have been investigated in continuous time by Börgers and Sarin[15], Hopkins[44], Coucheney et al.[28] and many others. The model closely related to the one discussed in this paper was proposed by Coucheney et al.[28]. In this article a class of penalty-regulated game dynamics consisting of replicator-like drift plus a penalty term that keeps agents from approaching the boundary of the state space is derived. This gives dynamics equivalent to the case when agents are scoring their actions by comparing their exponentially discounted cumulative payoffs over time and using a smooth best response to pick an action. They show global convergence in the continuous case.

Our model is also related to discrete-time model of Q-learning[44,50,73]. From a discrete-time viewpoint Leslie and Collins[50] used a Q-learning approach to establish the convergence of the resulting learning algorithm in two-player games under minimal information assumptions, a similar approach was also used by Cominetti et al.[27]. Moreover, the model presented in this article subsumes two well-known dynamics: a discrete-time variant of replicator dynamics (Multiplicative Weights Update)[34] and logit best-response[2,14]. Finally, it is a two-parameter version of EWA dynamics which was numerically studied for random (zero-sum) games by Galla and Farmer[36] and Pangallo et al.[62].

Chaos in games. The question one may want to answer is how complicated (or random) the behavior of agents may become even in simple games. The seminal work of Sato et al.[69] showed analytically that even in a simple two-player game of Rock-Paper-Scissor replicator dynamics (the continuous-time variant of MWU) can lead to chaos, rendering the equilibrium strategy inaccessible. Replicator dynamics has recently been shown to be able to produce arbitrarily complex orbits (e.g. Lorenz butterfly dynamics) in simple matrix games[4].

For two-player games with a large number of available strategies (complicated games), EWA algorithm, exhibits chaotic behavior in a large parameter space[36]. The prevalence of these chaotic dynamics also persists in games with many players, as shown in the follow-up work[67]. Careful examinations suggest a complex behavioral landscape in many games (small or large) for which no single theoretical framework currently applies. Moreover, a chaotic behavior was detected for Nash maps in games like match-

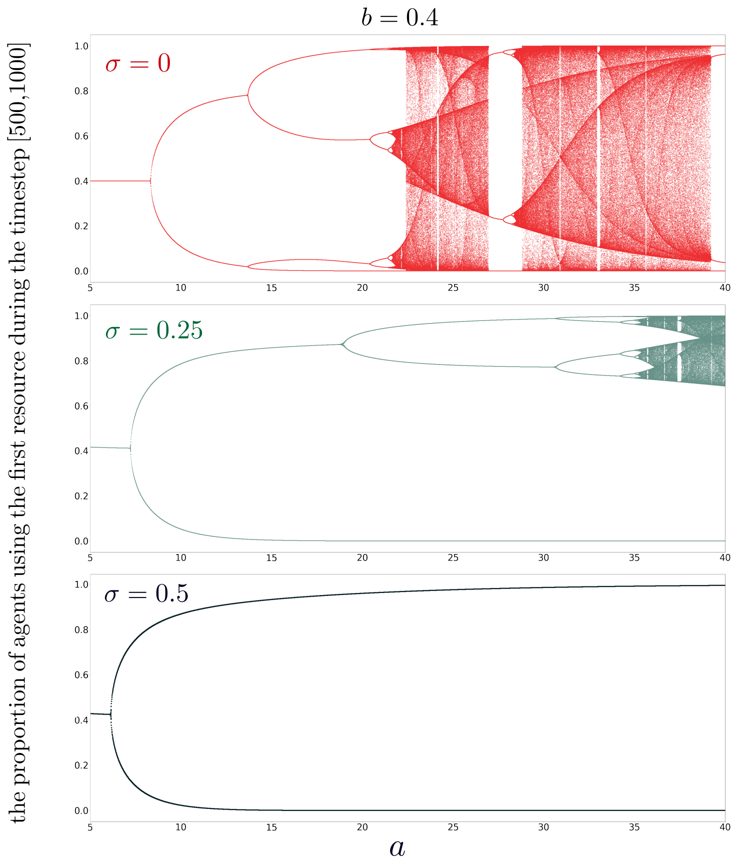


FIG. 1: Bifurcation diagrams for increasing values of the discount factor σ when the Nash equilibrium is fixed at $b = 0.4$. Without any discounting ($\sigma = 0$), the dynamics is described by the Multiplicative Weight Update which leads to the period-doubling bifurcation route to chaos as one increases the intensity of choice a [24]. However, as the discount factor increases to $\sigma = 0.25$, chaos starts at a much larger intensity of choice. As the discount factor increases to $\sigma = 0.5$, increasing the intensity of choice can at most lead to the period-2 dynamics. Thus, a larger discount factor tends to make the dynamics more stable and predictable.

ing pennies[7,37]. Fictitious play learning dynamics for a class of 3x3 games, including the Shapley's game and zero-sum dynamics, possesses rich periodic and chaotic behaviors[72,74]. Result that the replicator dynamics, the continuous-time variant of MWU, is Poincaré recurrent in zero-sum games[64], was later generalized[55] to Follow-the-Regularized-Leader (FTRL) algorithms (called also dual averaging). When MWU/FTRL is applied with constant step-size in zero-sum games it becomes unstable[5] and in fact Lyapunov chaotic[22]. It was showed experimentally[63] that EWA leads to limit cycles and high-dimensional chaos in two-agent games with negatively correlated payoffs. Lyapunov chaos was also established in the case of coordination/potential games for a variant of MWU, known as Optimistic MWU[23]. However, none of the above results implies formal chaos in the sense of Li-Yorke or positive topological entropy.

The first formal proof of Li-Yorke chaos was shown for MWU in a single instance of two agent two strategy congestion game by Palaiopanos et al.[61]. This result was generalized and strengthened (positive topological entropy) for all two-agent two-strategy coordination games[25]. In arguably the main precursor of our work[24] topological chaos in nonatomic congestion game where agents use MWU was established. This result was then extended to FTRL with steep regularizers[9]. The theory of Li-Yorke chaos has since then been applied in other game theoretic settings such as Cournot competition, Fisher markets and auctions[21], as well as blockchain protocols[49].

Finally, there is a growing landscape of one dimensional discrete time economic models. Most of those for which the dynamics can be thoroughly studied are those that can be described with a unimodal map[6,29,30,45,53,58]. In our model the interesting dynamics appears when map describing the game dynamics is bimodal (with two critical points). Such maps describe more complex systems with more complicated dynamics. As the theory of unimodal maps can be seen as complete, the same cannot be said about bimodal maps. Nevertheless, we show how one can study dynamics of such maps and for our family of maps we give precise description of the complicated dynamics.

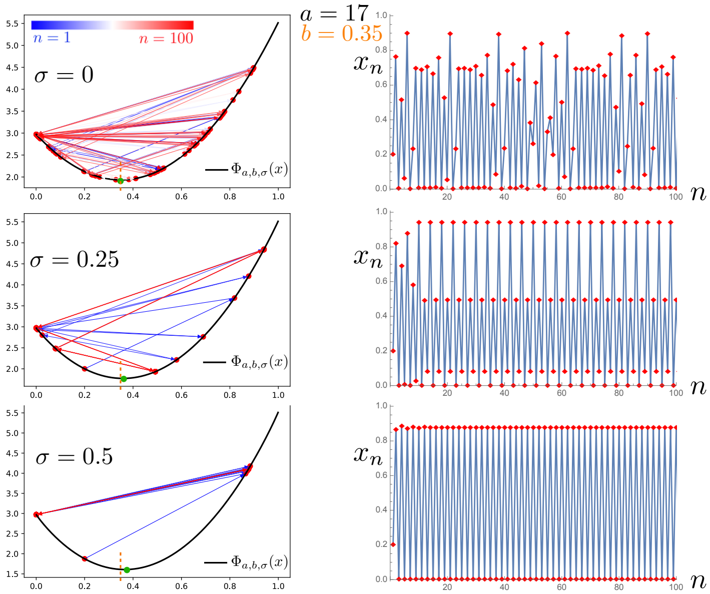


FIG. 2: A larger discount factor σ helps stabilize the dynamics. These figures are visualizations of the dynamics for $b = 0.35$ and fixed $a = 17$. The initial state is set to $x_0 = 0.2$. The left column shows the dynamics in the convex potential (cost) landscape in our congestion game $\Phi_{a,b,\sigma}(x) = \frac{a}{2}\left((1-b)x^2 + b(1-x)^2\right) + \sigma\left[x\log(x) + (1-x)\log(1-x)\right]$ whose quantal response equilibrium $\overline{x}$ is the potential minimum. This equilibrium (marked as green point) lays between Nash equilibrium $b = 0.35$ and 0.5. The right column shows the dynamics of x_n. Larger discount factors tend to stabilize the chaotic dynamics to at most a period 2 instability. The top row with $\sigma = 0$ corresponds to the Multiplicative Weights Update dynamics, whose time-average of x_n is exactly the Nash equilibrium b[24].

III. MODEL

We consider a two-strategy nonatomic *congestion game* (see[65]) with a continuum of agents (players), where all of them apply the *simplified Experience-Weighted Attraction* (sEWA) algorithm to update their strategies. Each of the agents controls an infinitesimal small fraction of the flow. The total flow of all the agents is equal to N. We will denote the fraction of the agents adopting the first strategy at time n as x_n.

A. Linear congestion games

The cost of each resource (path, link, route or strategy) here will be assumed proportional to the *load*. By denoting c_j the cost of selecting the strategy number j (when a fraction x of the agents choose the first strategy), if the coefficients of proportionality are $\alpha, \beta > 0$, we obtain

$$c_1(x) = \alpha N x, \qquad c_2(1-x) = \beta N(1-x). \qquad (1)$$

Without loss of generality we will assume throughout the paper that $\alpha + \beta = 1$. The values of α and $\beta = 1-\alpha$ indicate how different the resource costs are from each other. As we will see, the only parameter of the game that is important is the value of the equilibrium split, i.e. the fraction of agents using the first strategy at equilibrium. An advantage of this formulation is that the fraction of agents using each strategy at equilibrium is independent of the flow N. It is worth to add that our analysis on the emergence of bifurcations, limit cycles and chaos will carry over immediately to the cost functions of the form $\alpha x + \gamma$ for $\alpha, \gamma > 0$.

B. Learning in congestion games with sEWA

Experience-Weighted Attraction (EWA) has been proposed by Camerer and Ho[41] as a stochastic algorithm that binds reinforcement learning and belief learning algorithms. This unifying property comes with a consequence of many free parameters. In this paper we focus on a deterministic, simplified variant of EWA[36], which has only two free parameters, an intensity of choice and a memory loss parameter, which can be seen as the rate of exponential discounting of past costs.

We assume that at time $n+1$ the agents know the cost of the strategies at time n (equivalently, the fraction of agents using the first (x_n) and the second ($1-x_n$) strategy). Since we have a continuum of agents, the realized flow (split) is accurately described by the probabilities $(x_n, 1-x_n)$. There is a parameter $\varepsilon \in (0,1)$, which can be treated as the common learning rate of all agents, such that $\lambda = \log\frac{1}{1-\varepsilon}$ describes intensity of choice. Then the agents update their choices using sEWA algorithm

$$
\begin{aligned}
&x_{n+1} = \\
&\frac{x_n^{1-\sigma}\exp(-\lambda c_1(x_n))}{x_n^{1-\sigma}\exp(-\lambda c_1(x_n))+(1-x_n)^{1-\sigma}\exp(-\lambda c_2(1-x_n))} = \\
&\frac{x_n^{1-\sigma}}{x_n^{1-\sigma}+(1-x_n)^{1-\sigma}\exp[\lambda(c_1(x_n)-c_2(1-x_n))]} = \\
&\frac{x_n^{1-\sigma}}{x_n^{1-\sigma}+(1-x_n)^{1-\sigma}\exp(a(x_n-b))},
\end{aligned} \tag{2}
$$

where $a = N\log\frac{1}{1-\varepsilon} > 0$ is a population intensity of choice, $b = \beta \in (0,1)$ is the equilibrium split, i.e. the fraction of agents using the first strategy at equilibrium and $\sigma \in [0,1]$ is a memory loss parameter.

Note that if σ is treated as a discount factor then it describes how individuals value the past — the greater σ the less important are previous plays, the more important recent plays.

Observe that the update rule in (2) can be seen as Multiplicative Weights with perturbed costs. Let us consider cost functions perturbed by a component dependent on (exploration) parameter σ

$$
\begin{aligned}
\bar{c}_1(x) &= c_1(x) + \frac{\sigma}{\lambda}\log(x), \\
\bar{c}_2(1-x) &= c_2(1-x) + \frac{\sigma}{\lambda}\log(1-x).
\end{aligned} \tag{3}
$$

Then the dynamics introduced by (2) is Multiplicative Weights Update dynamics with perturbed costs functions (the precise derivation can be found in Appendix B).

Equation (2) implies that the dynamics of changes in the behavior of agents (and the system) is governed by the map

$$
f_{ab\sigma}(x) = \frac{x^{1-\sigma}}{x^{1-\sigma}+(1-x)^{1-\sigma}\exp(a(x-b))}, \tag{4}
$$

where $a > 0$, $b \in (0,1)$, $\sigma \in [0,1]$ (in this formula, when $\sigma = 1$, 0^0 is treated as 1).

C. Attracting orbits and chaos

Let us introduce basic notions of dynamical systems.

Definition III.1. *Let x be a fixed point of a dynamical system (X,f). The fixed point x is called:*

- *attracting, if there is an open neighborhood $U \subset X$ of x such that for every $y \in U$ we have $\lim_{n\to\infty} f^n(y) = x$, where f^n is a composition of the map f with itself n-times.*
- *repelling, if there is an open neighborhood $U \subset X$ of x such that for every $y \in U$, $y \neq x$ there exists $n \in \mathbb{N}$ such that $f^n(y) \in X \setminus U$.*

In this note we study differentiable maps on the unit interval. Then let x be a fixed point, if $|f'(x)| < 1$, then x is attracting, if $|f'(x)| > 1$, then x is repelling. If $|f'(x)| = 1$ we need more information.

Definition III.2. *Let (X,f) be a dynamical system. An orbit $\{f^n(x)\}$ is called periodic of period T if $f^{n+T}(x) = f^n(x)$ for any $n \in \mathbb{N}$. The smallest such T is called the period of x. The periodic orbit is called attracting, if x is an attracting fixed point of (X, f^T), and repelling, if x is a repelling fixed point of (X, f^T).*

There are many properties which can be seen as defining chaotic behavior. In this note we reflect on two of most commonly used ones: Li-Yorke chaos and positivity topological entropy.

Definition III.3 (Li-Yorke chaos)**.** *Let (X,f) be a dynamical system and $x,y \in X$. We say that (x,y) is a* Li-Yorke pair *if*

$$
\liminf_{n\to\infty} dist(f^n(x), f^n(y)) = 0
$$

and

$$
\limsup_{n\to\infty} dist(f^n(x), f^n(y)) > 0.
$$

A dynamical system (X,f) is Li-Yorke chaotic *if there is an uncountable set $S \subset X$ (called* scrambled set*) such that every pair (x,y) with $x,y \in S$ and $x \neq y$ is a Li-Yorke pair.*

The origin of the definition of Li-Yorke chaos is in the seminal Li and Yorke's article[52]. Intuitively orbits of two points from the scrambled set have to gather themselves arbitrarily close and spring aside infinitely many times but (if X is compact) it cannot happen simultaneously for each pair of points.

Another crucial feature of the chaotic behavior of a dynamical system is exponential growth of the number of distinguishable orbits. This happens if and only if the topological entropy of the system is positive. In fact positivity of topological entropy turned out to be an essential criterion of chaos[38]. This choice comes from the fact that the future of a deterministic (zero entropy) dynamical system can be predicted if its past is known[76] and positive entropy is related to randomness and chaos.

For every dynamical system over a compact phase space, we can define a number $h(f) \in [0,\infty]$ called the *topological entropy* of transformation f.

For a given positive integer n we define the n-th Bowen-Dinaburg metric on X, ρ_n^f as

$$
\rho_n^f(x,y) = \max_{0\le i<n} dist(f^i(x), f^i(y)).
$$

We say that the set E is (n,ε)-separated if $\rho_n^f(x,y) > \varepsilon$ for any distinct $x,y \in E$ and we denote by $s(n,\varepsilon,f)$ the cardinality of the most numerous (n,ε)-separated set for (X,f).

Definition III.4. *The topological entropy of f is defined as*

$$h(f) = \lim_{\varepsilon \searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon,f).$$

We give the intuitive explanation of the idea. Let us assume that we observe the dynamical system with the precision $\varepsilon > 0$, that is, we can distinguish any two points only if they are apart by at least ε. Then, after n iterations we will see at most $s(n,\varepsilon,f)$ different orbits. If transformation f is mixing points, then $s(n,\varepsilon,f)$ will grow. Taking upper limit over n will give us the asymptotic exponential growth rate of number of (distinguishable) orbits, and going with ε to zero will give us the quantity which can be treated as a measure of exponential speed, with which the number of orbits grow (with n). Thus, as Li-Yorke chaos tells us if there is chaos in the system, the topological entropy tells us *how much* of chaos we have. For deeper discussion we refer the reader to the excellent surveys by Blanchard[10], Glasner and Ye[39], Li and Ye[51] and Ruette's book[66].

D. Equilibrium

We assume that the population of agents is homogeneous, that is all agents use the same mixed strategy. Hence, adoption of a strategy profile $(x, 1-x)$ by agents results in x fraction of the agents choosing the first strategy. A strategy profile $(x, 1-x)$ is a *Nash (Wardrop) equilibrium* if and only if no agent can strictly decrease her expected cost by unilaterally deviating to another strategy. The interior fixed point of the dynamics introduced by (4) usually won't be a Nash equilibrium (see discussion in Section IV B). One can show that it agrees with general concept of *quantal response equilibrium.* Quantal response equilibrium (QRE)[54] was rediscovered in different contexts. In a quantal response equilibrium, the probability of playing action i for a given agent is proportional to $e^{-\theta\cdot c_i}$, where c_i is the expected cost of action i given that all other agents play according to the QRE. Namely, it is a bounded rationality version of Nash equilibrium where agents play suboptimal strategies with positive probability but where very costly errors are exponentially unlikely. In addition, as we have one to one correspondence between the ratio of agents choosing first resource in the game x and mixed strategy $(x, 1-x)$, we will use the simplification saying that the fixed point of $f_{ab\sigma}$ is a Nash/quantal response equilibrium of the game.

Take an interior fixed point $\overline{x}$ of $f_{ab\sigma}$. By (2) it must satisfy the follwing equations:

$$\log(\overline{x}) = (1-\sigma)\log(\overline{x}) + \log(1-\varepsilon)c_1(\overline{x}) - \log Z$$

and

$$\log(1-\overline{x}) = (1-\sigma)\log(1-\overline{x}) + \log(1-\varepsilon)c_2(1-\overline{x}) - \log Z,$$

where Z is the denominator in the first line of (2). From the above system of equations we derive that

$$\frac{N\sigma}{a}\log\frac{\overline{x}}{1-\overline{x}} = c_2(1-\overline{x}) - c_1(\overline{x}). \tag{5}$$

Thus, the probability of playing action i is proportional to $e^{-\theta\cdot c_i}$ where $\theta = \frac{a}{N\sigma}$. Equation (5) shows that when $\sigma = 0$ the fixed point $\overline{x}$ is a Nash equilibrium, whereas $\sigma > 0$ perturbs the equilibrium state. We use this interpretation in Section IV D, where we discuss perturbed costs.

IV. RESULTS

A. One dimensional map and topological conjugacy

We are interested in discrete dynamical system on the unit interval $[0,1]$:

$$x_{n+1} = f_{ab\sigma}(x_n) = \frac{x_n^{1-\sigma}}{x_n^{1-\sigma} + (1-x_n)^{1-\sigma}\exp(a(x_n-b))}, \tag{6}$$

where $a > 0$, $b \in (0,1)$, $\sigma \in [0,1]$.

To study dynamics of (6) one can look at the derivative of $f_{ab\sigma}$ for $x \in (0,1)$, which is given by

$$f'_{ab\sigma}(x) = \frac{x^{-\sigma}(1-x)^{-\sigma}\exp(a(x-b))\,[1-\sigma-ax(1-x)]}{\left[x^{1-\sigma} + (1-x)^{1-\sigma}\exp(a(x-b))\right]^2}, \tag{7}$$

or equivalently by

$$f'_{ab\sigma}(x) = f_{ab\sigma}(x)\big(1-f_{ab\sigma}(x)\big)\left(\frac{1-\sigma}{x(1-x)} - a\right). \tag{8}$$

From (8) when $\sigma < 1$ the map $f_{ab\sigma}$ is a homeomorphism as long as $a \in (0, 4(1-\sigma)]$, and when $a > 4(1-\sigma)$ the map $f_{ab\sigma}$ is N-bimodal with critical points (see Figure 3)

$$\kappa_l = \frac{1-\sqrt{1-\frac{4(1-\sigma)}{a}}}{2}, \qquad \kappa_r = \frac{1+\sqrt{1-\frac{4(1-\sigma)}{a}}}{2}.$$

Often instead of $f_{ab\sigma}$ we will consider another map $F\colon \mathbb{R} \mapsto \mathbb{R}$, conjugate to $f_{ab\sigma}|_{(0,1)}$. We get it by taking $y = \frac{1}{a}\log\frac{1-x}{x}$ and $F(y) = \frac{1}{a}\log\frac{1-f_{ab\sigma}(x)}{f_{ab\sigma}(x)}$. Then

$$F(y) = (1-\sigma)y + \frac{1}{e^{ay}+1} - b. \tag{9}$$

The map $x \mapsto \frac{1}{a}\log\frac{1-x}{x}$ is a diffeomorphism from $(0,1)$ onto $\mathbb{R}$. Therefore, $f_{ab\sigma}$ on $(0,1)$ is smoothly conjugate to F on $\mathbb{R}$.This means that instead of investigating the dynamics of $f_{ab\sigma}$, we may investigate the dynamics of F. It is worth adding that studying the dynamics of F is usually simpler than for $f_{ab\sigma}$. Thus, we will repeatedly look at the dynamics of our system through the lenses of the map F.

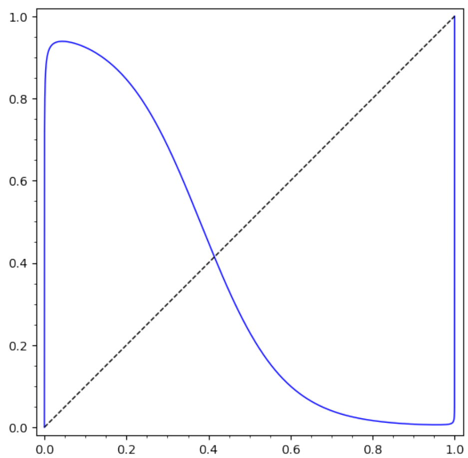


FIG. 3: Graph of $f_{ab\sigma}$ for $a = 12$, $b = 0.4$, $\sigma = 0.5$. The map is N-bimodal, that is, it has two critical points and $f(0) = 0$, $f(1) = 1$.

We have

$$F'(x) = 1 - \sigma - \frac{ae^{ax}}{(e^{ax}+1)^2}. \tag{10}$$

So, if $\sigma < 1$ then F is strictly increasing whenever $a \leq 4(1-\sigma)$; otherwise it is bimodal, with F increasing on the left and right laps, and decreasing on the middle lap.

The first example of the usefulness of F is finding the fixed points. For $\sigma = 1$ the map $f_{ab\sigma}$ is decreasing and has one equilibrium $\overline{x} \in (0,1)$. When $\sigma \in [0,1)$, we can check straightforwardly that 0 and 1 are fixed points of $f_{ab\sigma}$. To find the interior fixed points, we look at F. If y is sufficiently large, then $F(-y) > -y$ and $F(y) < y$. Therefore, F has a fixed point. Since $F' < 1-\sigma < 1$, by the Mean Value Theorem, F cannot have two distinct fixed points. We will denote the unique fixed point of F by $\overline{y}$. Then $f_{ab\sigma}$ has a unique fixed point $\overline{x}$ in $(0,1)$ and $\overline{y} = \frac{1}{a}\log\frac{1-\overline{x}}{\overline{x}}$.

Thus, for $\sigma \in [0,1)$ the map $f_{ab\sigma}$ has three equilibria: 0, 1, and $\overline{x} \in (0,1)$. The unique equilibrium in $(0,1)$ usually depends on a,b and σ (we will focus on its properties in the next section). When $\sigma \in (0,1)$, the derivative of $f_{ab\sigma}$ is infinite at 0 and 1, while for $\sigma = 0$ it is greater than 1 for both of these points. Therefore, the fixed points 0 and 1 are repelling independently of the values of intensity of choice $a > 0$, discount factor $\sigma \in [0,1)$ and Nash equilibrium of the game $b \in (0,1)$.

The map F has some nice properties, for instance it has negative Schwarzian derivative for $a > 4(1-\sigma)$. This is important as the dynamics is fairly regular if the map has negative Schwarzian derivative. Recall that the Schwarzian derivative of a map f having third derivative is given by

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

Lemma IV.1. *If $a > 4(1-\sigma)$ then the Schwarzian derivative of F is negative.*

Proof. For simplicity, let us use notation $t = e^{ax}$. Elementary calculations give us

$$F'(x) = 1-\sigma - a\frac{t}{(1+t)^2},$$
$$F''(x) = a^2\frac{t^2-t}{(1+t)^3},$$
$$F'''(x) = a^3\frac{-t^3+4t^2-t}{(1+t)^4}.$$

Schwarzian derivative of F is negative if and only if $2F'F''' - 3(F'')^2 < 0$. From our formulas we get

$$2F'F''' - 3(F'')^2 = \big(2(1-\sigma)(-t^2+4t-1) - at\big)\frac{a^3t}{(1+t)^4}.$$

We have

$$\begin{aligned} &2(1-\sigma)(-t^2+4t-1) - at = 2(1-\sigma)\big(2t-(t-1)^2\big) - at \\ &\leq 4(1-\sigma)t - at = (4(1-\sigma)-a)t, \end{aligned}$$

so if $a > 4(1-\sigma)$ then $SF < 0$. □

Observe that if F isn't monotone, then it has negative Schwarzian derivative.

B. Properties of the interior equilibrium

As the fixed points 0 and 1 are always repelling, we focus our attention on $\overline{x}$ — the unique fixed point of the dynamics (6) inside the unit interval. We now describe the location of the interior fixed point $\overline{x}$ (QRE) and its monotonic convergence to either 1/2 or Nash equilibrium b.

Proposition IV.2. *Let $\sigma \in (0,1]$, and let $\overline{x} \in (0,1)$ be a (unique) fixed point of $f_{ab\sigma}$. Then $\overline{x}$ lies between 1/2 and b. Moreover, when the intensity of choice a tends to zero, $\overline{x}$ tends monotonically to 1/2, while as intensity of choice tends to infinity, $\overline{x}$ converges monotonically to Nash equilibrium b. For $\sigma = 0$, $\overline{x} = b$ is the unique Nash equilibrium for all $a > 0$.*

Proof. We start by noticing that the equality $f_{ab\sigma}(\overline{x}) = \overline{x}$ is equivalent to

$$\overline{x} = b + \frac{\sigma}{a}\log\left(\frac{1-\overline{x}}{\overline{x}}\right), \tag{11}$$

The first assertion follows from the fact that, by (11), $\overline{x} \geq b$ if and only if $\overline{x} < \frac{1}{2}$, and $\overline{x} < b$ otherwise.

We proceed with the second assertion. We put (11) in the equivalent form

$$a(\overline{x}-b) = \sigma\log\left(\frac{1-\overline{x}}{\overline{x}}\right). \tag{12}$$

Denote the left-hand side of (12) by $L(a)$, and the right-hand side of (12) by $R(a)$. Note that $\lim_{a\to 0^+} L(a) = 0$. By (12), $\lim_{a\to 0^+} R(a) = 0$. Since $\sigma > 0$, it follows that $\lim_{a\to 0^+} \overline{x} = \frac{1}{2}$.

We claim that if $b \in (0,1/2)$, then $\frac{d\overline{x}}{da} < 0$, and if $b \in (1/2,1)$, then $\frac{d\overline{x}}{da} > 0$. Rewrite (11) as

$$a\overline{x} = ab + \sigma \log \frac{1-\overline{x}}{\overline{x}}.$$

Take the total derivative of both sides with respect to a. We get

$$\overline{x} + a\frac{d\overline{x}}{da} = b - \frac{\sigma}{\overline{x}(1-\overline{x})} \cdot \frac{d\overline{x}}{da}.$$

Therefore,

$$\left(a + \frac{\sigma}{\overline{x}(1-\overline{x})}\right)\frac{d\overline{x}}{da} = b - \overline{x}. \tag{13}$$

If $b \in (0,1/2)$, then $b - \overline{x} < 0$; if $b \in (1/2,1)$ then $b - \overline{x} > 0$. This completes the proof of our claim.

It follows that $\overline{x}$ varies monotonely with a. In particular, $\lim_{a\to\infty} R(a)$ exists. Since $\overline{x}$ is between b and $1/2$, this limit is finite. Therefore, $\lim_{a\to\infty} L(a)$ exists and is finite, so $\lim_{a\to\infty} \overline{x} = b$.

The case of $\sigma = 0$ follows from (11). □

Remark IV.3. *From the proof above it follows that if $b \neq 1/2$ then $(1-2\overline{x})\frac{d\overline{x}}{da} < 0$.*

Proposition IV.2 guarantees that the fixed point is bounded by $\frac{1}{2}$ and b. Moreover, it describes two extreme cases. When the intensity of choice tends to zero, the fixed point $\overline{x}$ approaches the case when an agent is indifferent about her payoff and thus which resource to choose. As both choices are equally likely, the split $(\frac{1}{2}, \frac{1}{2})$ is chosen. On the other hand, if intensity of choice tends to infinity, then a small historical advantage of a given choice causes that choice to be more probable. Then the fixed point (QRE) approaches Nash equilibrium.

We next study possible convergence of trajectories of the dynamics (4) to the fixed point. Proposition IV.2 implies that when $\sigma > 0$ the fixed point will approach Nash equilibrium for sufficiently large values of a. Thus, one may be interested in choosing large values of the parameter a. But will the dynamics stabilize close to Nash equilibrium? We start with lemma about the map F.

Lemma IV.4. *If the trajectories of all points $x < \overline{y}$ are attracted to $\overline{y}$, then the trajectories of all points of $\mathbb{R}$ are attracted to $\overline{y}$. Similarly, if the trajectories of all points $x > \overline{y}$ are attracted to $\overline{y}$, then the trajectories of all points of $\mathbb{R}$ are attracted to $\overline{y}$.*

Proof. Assume that there is a point of $\mathbb{R}$, whose trajectory is not attracted to $\overline{y}$. Since both $-\infty$ and ∞ are repelling, by[70], F has a periodic orbit of period 2. If the trajectories of all points $x < \overline{y}$ (respectively, $x > \overline{y}$) are attracted to $\overline{y}$, this periodic orbit has to lie entirely to the right (respectively, left) of $\overline{y}$. Thus, there is a fixed point to the right (respectively, left) of $\overline{y}$, a contradiction. □

We show that although for small values of intensity of choice we see global convergence to QRE, increasing the intensity of choice will result in losing stability of $\overline{x}$, and thus, the system will become unstable.

Theorem IV.5.

1. *If $\sigma \in [0,1]$, then as long as $\overline{x}$ is attracting for $f_{ab\sigma}$, it attracts all trajectories of points from $(0,1)$. For $\sigma = 1$ equilibrium $\overline{x}$ attracts also trajectories of 0 and 1.*

2. *For any $b \in (0,1)$ there exists $a_0 > 0$ such that $\overline{x}$ is attracting for $a < a_0$, and $\overline{x}$ is repelling for $a > a_0$. Moreover, the threshold a_0 is decreasing with respect to the exploration parameter σ.*

Proof. We begin with the first assertion. We will show that if the fixed point of F is attracting, then it is globally attracting, which by the conjugacy argument will prove our theorem.

Let $\sigma \in [0,1)$. If F is strictly increasing, then it does not have a periodic orbit of period 2, so $\overline{y}$ is globally attracting.

Assume that F is bimodal. If $\overline{y}$ belongs to the left or right lap, then by Lemma IV.4, $\overline{y}$ is globally attracting. Assume that $\overline{y}$ belongs to the interior of the middle lap. Since by Lemma IV.1 the Schwarzian derivative of F is negative, then the interval joining $\overline{y}$ with one of the critical points of F is in the basin of attraction A of $\overline{y}$. We may assume that this critical point is the left one, κ_-. There is a unique point $y < \kappa_-$ such that $F(y) = \overline{y}$. Then $F([y,\kappa_-]) = F([\kappa_-,\overline{y}]) \subset A$, so $[y,\overline{y}] \subset A$. For every point $x < y$ we have $x < F(x) < \overline{y}$. Therefore, the trajectory of x increases as long as it stays to the left of y. Since there are no fixed points to the left of y, the trajectory has to enter $[y,\overline{y}]$ sooner or later. This proves that $(-\infty,\overline{y}] \subset A$, so by Lemma IV.4, $\overline{y}$ is globally attracting.

When $\sigma = 1$, then F is strictly decreasing. Values of F are bounded by $-b$ and $1-b$, so F has an attracting invariant interval $[-b,1-b]$. Because from Lemma IV.1 F has negative Schwarzian derivative and F has no critical points then, by Singer theorem, $(-\infty,\overline{y}]$ or $[\overline{y},\infty)$ has to be attracted by $\overline{y}$. By Lemma IV.4, $\overline{y}$ has to be globally attracting.

Now, we proceed with the proof of the second assertion. First we show that if $b \in (0,1)$, then $f'_{ab\sigma}(\overline{x})$ is decreasing as a function of a.

Assume first that $b \neq 1/2$. Multiply both sides of (13) by $1-2\overline{x}$:

$$(1-2\overline{x})\left(a + \frac{\sigma}{\overline{x}(1-\overline{x})}\right)\frac{d\overline{x}}{da} = (1-2\overline{x})(b-\overline{x}) =$$
$$b(1-b) + (b-\overline{x})^2 - \overline{x}(1-\overline{x}).$$

From this and the fact that in view of (8) we have

$$f'_{ab\sigma}(\overline{x}) = 1-\sigma - a\overline{x}(1-\overline{x}), \tag{14}$$

we get

$$\frac{df'_{ab\sigma}(\overline{x})}{da} = -\overline{x}(1-\overline{x}) - a(1-2\overline{x})\frac{d\overline{x}}{da} = \frac{\sigma}{\overline{x}(1-\overline{x})}(1-2\overline{x})\frac{d\overline{x}}{da} - b(1-b)-(b-\overline{x})^2.$$

In view of Remark IV.3, this is negative.

If $b=1/2$, then $\overline{x}=1/2$, and $f'_{ab\sigma}(\overline{x}) = 1-\sigma-a/4$, so also $f'_{ab\sigma}(\overline{x})$ is decreasing as a function of a. Thus, the threshold a_0 is unique. Moreover, by the first assertion of our theoremas long as $\overline{x}$ is attracting it attracts all points from $(0,1)$.

We now show that the threshold a_0 is decreasing with respect to σ.

Let $0 \le \sigma_1 < \sigma_2 \le 1$. Then

$$\begin{aligned} f_{ab\sigma_1}(x) < f_{ab\sigma_2}(x) \iff x < \frac{1}{2}, \\ f_{ab\sigma_1}(x) > f_{ab\sigma_2}(x) \iff x > \frac{1}{2}. \end{aligned} \tag{15}$$

These inequalities follow from the fact that

$$f_{ab\sigma_1}(x) < f_{ab\sigma_2}(x) \;\Leftrightarrow\; \left(\frac{1-x}{x}\right)^{\sigma_2-\sigma_1} > 1,$$

$$f_{ab\sigma_1}(x) > f_{ab\sigma_2}(x) \;\Leftrightarrow\; \left(\frac{1-x}{x}\right)^{\sigma_2-\sigma_1} < 1.$$

Let $b<1/2$. Then $\overline{x} \in [b,1/2]$. From (15) we have

$$f_{ab\sigma_2}(\overline{x}_{\sigma_1}) > f_{ab\sigma_1}(\overline{x}_{\sigma_1}) = \overline{x}_{\sigma_1}.$$

Thus, once more from (15), we infer that $1/2 \ge \overline{x}_{\sigma_2} > \overline{x}_{\sigma_1} \ge b$.

Therefore, from (14) and the fact that a term $z(1-z)$ increases if and only if the distance between z and $\frac{1}{2}$ decreases, we have $f'_{ab\sigma_1}(\overline{x}_{\sigma_1}) > f'_{ab\sigma_2}(\overline{x}_{\sigma_2})$ for any given $a>0$. Similar reasoning can be performed for $b>1/2$. Since for $b=1/2$ the only difference in the reasoning is that $\overline{x}_{\sigma_1} = \overline{x}_{\sigma_2} = 1/2$, we obtain that for every $b \in (0,1)$ and $a>0$

$$f'_{ab\sigma_1}(\overline{x}_{\sigma_1}) > f'_{ab\sigma_2}(\overline{x}_{\sigma_2}).$$

Thus (as the derivative at the fixed point cannot be greater than one), the instability at $\overline{x}_{\sigma_2}$ arises for smaller values of a than for $\overline{x}_{\sigma_1}$. □

Theorem IV.5 implies that for small intensity of choice the equilibrium attracts all trajectories of the system, so starting from any initial state (other than the case where the entire population chooses a pure strategy) the system will converge to $(\overline{x}, 1-\overline{x})$. Therefore, the description of the dynamics of the game is simple as long as $\overline{x}$ is attracting (It is worth mentioning that existence of attracting quantal response equilibrium or even attracting Nash equilibrium does not exclude possibility of chaotic behavior. For instance, Follow-the-Regularized-Leader algorithm admits coexistence of an attracting Nash equilibrium and chaos[9]). Nevertheless, there is a threshold where the equilibrium loses stability. Therefore, increasing intensity of choice will eventually destabilize the system. This threshold depends on discount factor σ in a monotonic way — as we value recent costs more (σ increases), the instability appears earlier, for smaller intensity of choice.

Proposition IV.6. *If $\sigma>0$, then there exists a threshold $a^*>0$ such that for any $a>a^*$ the fixed point $\overline{x}$ is repelling for every $b\in(0,1)$. If $\sigma=0$, then for any intensity of choice a there exists b (sufficiently close to 0 or 1) such that the Nash equilibrium $\overline{x}$ is attracting.*

Proof. Let us think of the picture in the (a,b)-plane, see Figure 4. The region of stability of the fixed point is marked in yellow, and the region with an attracting periodic orbit of period 2 in red. The curve dividing those two regions corresponds to the first period doubling bifurcation, where the fixed point loses stability. Thus, for (a,b) on this curve the derivative of $f_{ab\sigma}$ at the fixed point $\overline{x}$ is equal to -1.

To find this curve, we use formula (8) with $x=\overline{x}$. We get

$$f'_{ab\sigma}(\overline{x}) = \overline{x}(1-\overline{x})\left(\frac{1-\sigma}{\overline{x}(1-\overline{x})} - a\right) = 1-\sigma-a\overline{x}(1-\overline{x}).$$

Thus, the equation $f'_{ab\sigma}(\overline{x}) = -1$ is equivalent to

$$\overline{x}^2 - \overline{x} + \frac{2-\sigma}{a} = 0. \tag{16}$$

When $a \ge 4(2-\sigma)$, this equation has two solutions, symmetric with respect to $1/2$, namely

$$\overline{x}_1 = \frac{1}{2}\left(1-\sqrt{1-\frac{4(2-\sigma)}{a}}\right),$$

$$\overline{x}_2 = \frac{1}{2}\left(1+\sqrt{1-\frac{4(2-\sigma)}{a}}\right).$$

Moreover, with $\sigma\in[0,1]$ fixed, $\overline{x}_1$ as a function of a is a bijection from $[4(2-\sigma),\infty)$ onto $(0,\frac{1}{2}]$, and $\overline{x}_2$ is a bijection from $[4(2-\sigma),\infty)$ onto $[\frac{1}{2},1)$.

Since $\overline{x}_1$ and $\overline{x}_2$ are fixed points of $f_{ab\sigma}$, we can use formula (11), and we obtain formulas for b as functions of a and σ:

$$b_1(a,\sigma) = \overline{x}_1 - \frac{\sigma}{a}\log\left(\frac{1-\overline{x}_1}{\overline{x}_1}\right) \in \left[0,1/2\right],$$

$$b_2(a,\sigma) = \overline{x}_2 - \frac{\sigma}{a}\log\left(\frac{1-\overline{x}_2}{\overline{x}_2}\right) \in \left[1/2,1\right].$$

For a fixed σ, the first formula describes the bottom branch of our curve, and the second formula the upper branch. As $\overline{x}_2 = 1-\overline{x}_1$, we have $b_2(a,\sigma) = 1-b_1(a,\sigma)$.

We next find a solution of $b_1(a,\sigma)=0$ (and $b_2(a,\sigma)=1$). Since $\overline{x}_1$ and $\overline{x}_2$ are two solutions to (16), we have $\overline{x}_1\overline{x}_2=\frac{2-\sigma}{a}$. We get

$$\begin{aligned} b_1(a,\sigma)=0 &\iff \frac{\sigma}{a}\overline{x}_2\log\left(\frac{1-\overline{x}_1}{\overline{x}_1}\right)=\overline{x}_1\overline{x}_2 \\ &\iff (1-\overline{x}_1)\log\left(\frac{1-\overline{x}_1}{\overline{x}_1}\right)=\frac{2-\sigma}{\sigma}. \end{aligned}$$

Define

$$g(x)=(1-x)\log\left(\frac{1-x}{x}\right).$$

Then $g(1/2)=0$, $\lim_{x\to 0^+}g(x)=\infty$ and $g'(x)=-\log\left(\frac{1-x}{x}\right)-\frac{1}{x}<0$ for $x\in(0,1/2]$. Therefore, $g\colon (0,1/2]\mapsto[0,\infty)$ is bijection. By the fact that x_1 as a function of a is also a bijection, we obtain that for a fixed $\sigma\in(0,1]$ there exists a unique $a^*\in\big[4(2-\sigma),\infty\big)$ such that $b_1(a^*,\sigma)=0$ and $b_2(a^*,\sigma)=1$.

On the other hand, when $\sigma=0$, the equation $b_1(a,\sigma)=0$ does not have any solution. Instead

$$\lim_{a\to\infty}b_1(a,\sigma)=0, \quad \text{and} \quad \lim_{a\to\infty}b_2(a,\sigma)=1.$$

Therefore,

1. If $\sigma\in(0,1]$, then there exists $a^*\geq 4(2-\sigma)$ such that for every $a>a^*$ the fixed point $\overline{x}$ is repelling.
2. If $\sigma=0$, then for any $a>0$ there exists $b\in(0,1)$ (sufficiently close to 0 or sufficiently close to 1) such that the fixed point $\overline{x}=b$ is attracting.

This completes the proof. □

Proposition IV.6 gives another important distinction between no discount (full memory) model and discount (memory loss) case. When intensity of choice is large, then in full memory case one can change conditions of the game (differentiate costs of the (pure) strategies) to impose the convergence to equilibrium. However, once the memory loss affects choices of agents ($\sigma>0$), then for sufficiently large intensity of choice the system will inevitably become unstable and no change of conditions of the game will stabilize it.

After discussing what happens for small values of intensity of choice and formulating the stability loss conditions, in the next sections, which are a key part of the work, we will focus on the analysis of the case when the system is unstable.

C. Main result — memory-dependent behavior

In this section we discuss long-term behavior of agents for large intensity of choice when $\sigma\in[0,1]$. We study how behavior of the system for large values of intensity of choice a depends on the interplay between discount factor σ (property of learning) and difference in costs of resources (property of the game reflected by the value of b). Long-term behavior of agents differs when $b\in\left(\frac{1-\sigma}{2-\sigma},\frac{1}{2-\sigma}\right)$ and when $b\in\left(0,\frac{1-\sigma}{2-\sigma}\right)\cup\left(\frac{1}{2-\sigma},1\right)$, that is, it depends on proximity of the costs of resources (see Figure 4).

1. *Auxiliary lemmas*

First we show two simple lemmas which we will use in the proofs of main theorems. They give bounds for the conjugate map F.

Lemma IV.7. *For any $\varepsilon>0$ there exists $\alpha(\varepsilon)$ such that if $a\geq\alpha(\varepsilon)$ and $|x|\geq\varepsilon$ then $0\leq F'(x)<1-\sigma$.*

Proof. From (10) we have $F'(x)<1-\sigma$. Since $\frac{e^{ax}}{(e^{ax}+1)^2}<e^{ax}$, we get $F'(x)>1-\sigma-ae^{ax}$. Similarly, $\frac{e^{ax}}{(e^{ax}+1)^2}<e^{-ax}$, so $F'(x)>1-\sigma-ae^{-ax}$. Therefore, $F'(x)>1-\sigma-ae^{-a|x|}$. If $|x|\geq\varepsilon$, then we get $F'(x)>1-\sigma-ae^{-\varepsilon a}$. Since $\lim_{a\to\infty}ae^{-\varepsilon a}=0$, the lemma follows. □

Let us consider two linear maps,

$$F_-(x)=(1-\sigma)x+1-b \quad \text{and} \quad F_+(x)=(1-\sigma)x-b.$$

Since $0<\frac{1}{e^{ax}+1}<1$, we have $F_+<F<F_-$. Let us improve those estimates.

Lemma IV.8. *If $x<0$ then $F_-(x)+\frac{1}{ax}<F(x)<F_-(x)$, and if $x>0$ then $F_+(x)<F(x)<F_+(x)+\frac{1}{ax}$.*

Proof. If $x<0$ then

$$\begin{aligned} F_-(x)-F(x)=1-\frac{1}{e^{ax}+1}&=\frac{e^{ax}}{e^{ax}+1}<e^{ax} \\ &=\frac{1}{e^{-ax}}<\frac{1}{-ax}. \end{aligned}$$

If $x>0$ then

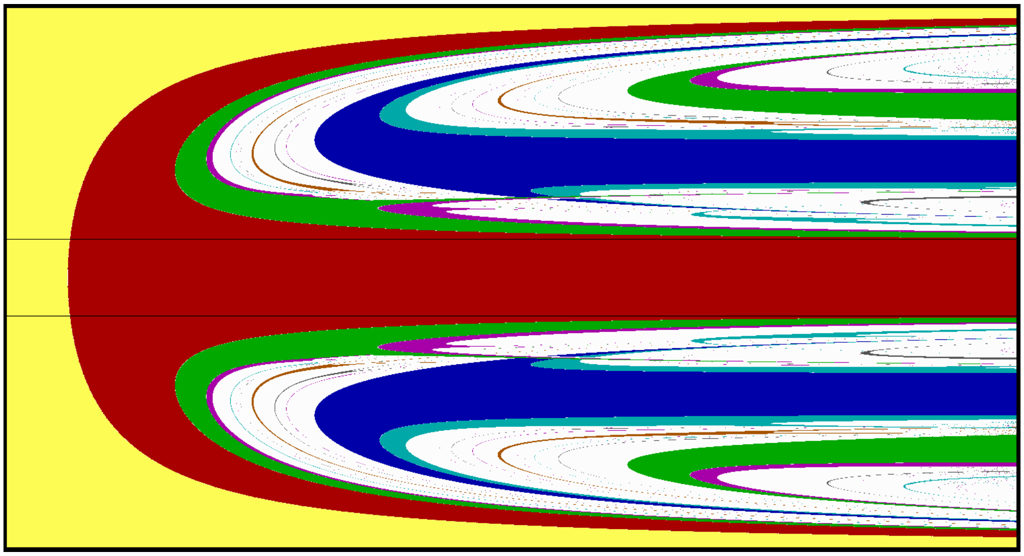
$$F(x)-F_+(x)=\frac{1}{e^{ax}+1}<\frac{1}{e^{ax}}<\frac{1}{ax}.$$

□

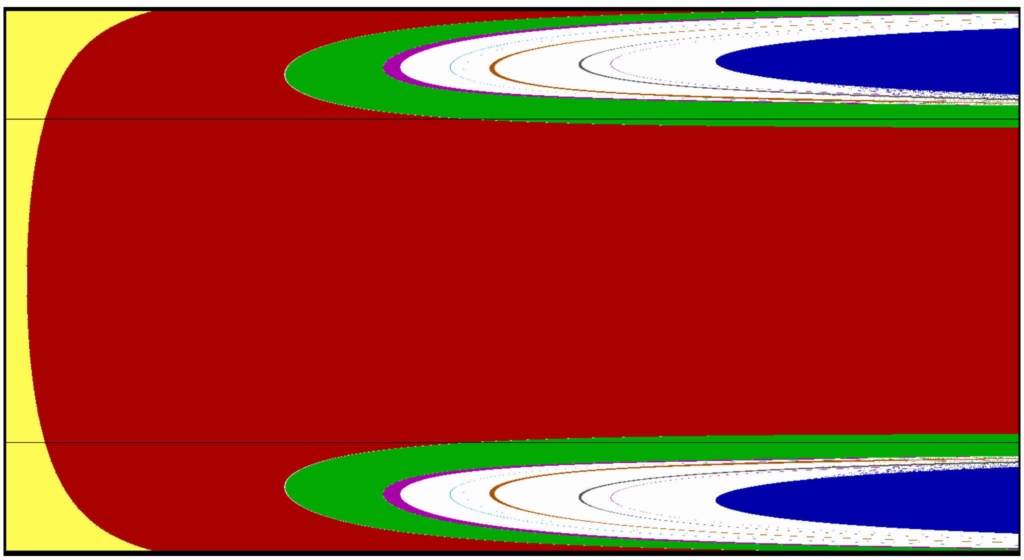
2. *Predictability for* $b\in\left(\frac{1-\sigma}{2-\sigma},\frac{1}{2-\sigma}\right)$

We begin with the case $b\in\left(\frac{1-\sigma}{2-\sigma},\frac{1}{2-\sigma}\right)$. We will investigate the existence of an attracting periodic orbit of period 2 when intensity of choice is large.

Theorem IV.9. *Let $\sigma\in[0,1]$ be fixed. For a given $b_0\in\left(\frac{1-\sigma}{2-\sigma},\frac{1}{2}\right)$, there exists $a_1>0$ such that if $a\geq a_1$ and $b\in[b_0,1-b_0]$ then $f_{ab\sigma}$ has an attracting periodic orbit of period 2 which attracts trajectories of all points from $(0,1)$, except countably many whose trajectories fall into $\overline{x}$.*



(a) $\sigma = 0.25$



(b) $\sigma = 0.75$

FIG. 4: Period diagrams of the small-period attracting periodic orbits associated with the map $f_{ab\sigma}$ for different values of σ. The horizontal axes are the intensity of choice $a \in [4, 54]$ and the vertical axes are the asymmetry of cost $b \in [0, 1]$. The colors encode the periods of attracting periodic orbits as follows: period 1 (fixed point) = yellow, period 2 = red, period 3 = blue, period 4 = green, period 5 = brown, period 6 = cyan, period 7 = darkgray, period 8 = magenta, and period larger than 8 = white. The equilibrium analysis is only viable when the fixed point is stable. In other region of the phase-space, non-equilibrating dynamics arise and system proceeds through the period-doubling bifurcation route to chaos in the white region. The picture is generated from the following algorithm: 20000 preliminary iterations are discarded. Then a point is considered periodic of period n if $|f^n_{ab\sigma}(x) - x| < 10^{-16}$ and it is not periodic of any period smaller than n. Black lines describe bounds $b = \frac{1-\sigma}{2-\sigma}$, $b = \frac{1}{2-\sigma}$. By Theorem IV.9 we know what happens for large a. We can see from numerical computations that situation might be a little bit more complicated for some values of a. There is a possibility of the attracting periodic orbit of period 4.

Corollary IV.10. *For a fixed $\sigma \in [0, 1]$ and $b \in \left(\frac{1-\sigma}{2-\sigma}, \frac{1}{2-\sigma}\right)$ there exists $a_1 > 0$ such that for $a \geq a_1$ trajectories of all points from $(0, 1)$ are attracted to the periodic orbit of period 2, except countably many whose trajectories fall into the interior equilibrium $\overline{x}$.*

The proof of Theorem IV.9 relies on careful choice of two disjoint intervals I_- and I_+ such that $F(I_-) \subset I_+$ and $F(I_+) \subset I_-$. This last property implies existence of an attracting periodic orbit of period 2. As we deal with a discrete dynamical system we have to take into account that some trajectories may fall into the repelling equilibrium. The property of attraction of almost all trajectories follows from existence of an attracting invariant set. We show that all trajectories eventually enter the invariant set and then they either hit the fixed point and stay there, or they are attracted by the periodic orbit.

Proof of Theorem IV.9. First, let us consider the case of $\sigma = 1$. From (9) the map

$$F(x) = \frac{1}{e^{ax}+1} - b$$

is decreasing. Thus, F^2 is increasing. This excludes existence of periodic orbits (other than fixed points) of F^2. As a result, F does not have any periodic orbit of period greater than 2. Thus, all trajectories converge to the fixed point $\overline{y}$ or a periodic orbit of period 2 of F.

Let a_0 be a threshold from Proposition 2. If $a < a_0$, then by Theorem IV.5, $\overline{y}$ is globally attracting. Take $a > a_0$. Notice that

$$\begin{aligned} F^2(x) = x \quad &\iff \quad \frac{1}{1+\exp(aF(x))} - b = x \\ &\iff \quad F(x) = \frac{1}{a}\log\left(\frac{1}{x+b} - 1\right). \end{aligned}$$

Therefore, if $\{\gamma_1, \gamma_2\}$ is a periodic orbit of period 2, then

$$\gamma_2 = \frac{1}{a}\log\left(\frac{1}{\gamma_1+b} - 1\right). \tag{17}$$

Assume that F has two attracting orbits of period 2: $\{\gamma_1', \gamma_2'\}$ and $\{\gamma_1'', \gamma_2''\}$. Without loss of generality we can assume that $\gamma_1' < \gamma_1''$. Then, by (17), we get that $\gamma_2' > \gamma_2''$. Since by Lemma IV.1 the Schwarzian derivative of F is negative, then in the immediate basin of attraction of each periodic orbit has to be $-b$ or $1-b$. We may assume that $[-b, \gamma_1')$ is in the basin of attraction of $\{\gamma_1', \gamma_2'\}$ and $(\gamma_2'', 1-b]$ is in the basin of attraction of $\{\gamma_1'', \gamma_2''\}$. But then γ_2' is attracted to $\{\gamma_1'', \gamma_2''\}$. This contradicts existence of two attracting periodic orbits.

Now, assume that $\sigma \in [0,1)$. Fix $b_0 \in \left(\frac{1-\sigma}{2-\sigma}, \frac{1}{2}\right)$ and assume that $b \in [b_0, 1-b_0]$. Set $\phi(b) = 2b - \sigma b - 1 + \sigma$. Note that $\phi(1-b) = 1 - 2b + \sigma b$. Since $b > b_0$ and $1-b > b_0$, we have

$$\phi(b) \geq \phi(b_0) > 0 \quad \text{and} \quad \phi(1-b) \geq \phi(b_0) > 0. \tag{18}$$

Set

$$x_- = -\frac{\phi(b)}{\sigma(2-\sigma)} \quad \text{and} \quad x_+ = \frac{\phi(1-b)}{\sigma(2-\sigma)}.$$

Observe that $x_- < 0 < x_+$. We have

$$\begin{aligned} &F_-(x_-) = (1-\sigma)\frac{-2b+\sigma b+1-\sigma}{\sigma(2-\sigma)} + (1-b) = \\ &\frac{1-2\sigma+\sigma^2-2b+3\sigma b-\sigma^2 b+2\sigma-\sigma^2-2b\sigma+\sigma^2 b}{\sigma(2-\sigma)} = \\ &\frac{1-2b+b\sigma}{\sigma(2-\sigma)} = x_+, \end{aligned}$$

and

$$\begin{aligned} &F_+(x_+) = (1-\sigma)\frac{1-2b+\sigma b}{\sigma(2-\sigma)} - b = \\ &\frac{1-2b+\sigma b-\sigma+2\sigma b-\sigma^2 b-2b\sigma+b\sigma^2}{\sigma(2-\sigma)} = \\ &\frac{1-2b+\sigma b-\sigma}{\sigma(2-\sigma)} = x_-. \end{aligned}$$

Thus,

$$F_-(x_-) = x_+ \text{ and } F_+(x_+) = x_-. \tag{19}$$

Set $K = \frac{\phi(b_0)}{2\sigma(2-\sigma)}$ and consider intervals $I_- = [x_- - K, x_- + K]$ and $I_+ = [x_+ - K, x_+ + K]$.

We have $|x_-| = \frac{\phi(b)}{\sigma(2-\sigma)}$, so by (18), for every $x \in I_-$ we get

$$|x| \geq \frac{\phi(b)}{\sigma(2-\sigma)} - K \geq K.$$

Similarly, $|x_+| = \frac{\phi(1-b)}{\sigma(2-\sigma)}$, so for every $x \in I_+$ we get

$$|x| \geq \frac{\phi(1-b)}{\sigma(2-\sigma)} - K \geq K.$$

So $|x| \geq K$ for all $x \in I_- \cup I_+$. Set $a_0 = \alpha(K)$. Then, by Lemma IV.7, if $a \geq a_0$ then

$$0 \leq F'(x) < 1 - \sigma \text{ for all } x \in I_- \cup I_+. \tag{20}$$

From (18), (19), (20), and Lemma IV.8, we have

$$\begin{aligned} F(I_-) &\subset [F(x_-) - (1-\sigma)K, F(x_-) + (1-\sigma)K] \\ &\subset \left[x_+ + \frac{1}{ax_-} - (1-\sigma)K, x_+ + (1-\sigma)K\right]. \end{aligned}$$

We have

$$\frac{1}{ax_-} - (1-\sigma)K = -\frac{\sigma(2-\sigma)}{a\phi(b)} - (1-\sigma)K \geq -K$$

for $a \geq \frac{2-\sigma}{K\phi(b)}$, so

$$F(I_-) \subset [x_+ - K, x_+ + (1-\sigma)K] \subset I_+.$$

Similarly,

$$\begin{aligned} F(I_+) &\subset [F(x_+) - (1-\sigma)K, F(x_+) + (1-\sigma)K] \\ &\subset \left[x_- - (1-\sigma)K, x_- + \frac{1}{ax_+} + (1-\sigma)K\right]. \end{aligned}$$

We have

$$\frac{1}{ax_+} + (1-\sigma)K \leq K,$$

for $a \geq \frac{2-\sigma}{K\phi(1-b)}$, so

$$F(I_+) \subset [x_- - (1-\sigma)K, x_- + K] \subset I_-.$$

Thus, for

$$a \geq a_1 = \max\left\{a_0, \frac{2-\sigma}{K\phi(b)}, \frac{2-\sigma}{K\phi(1-b)}\right\}$$

we have

$$F(I_-) \subset I_+ \text{ and } F(I_+) \subset I_-. \tag{21}$$

We may additionally assume that $a_1 > 4(1-\sigma)$. Then for $a \geq a_1$ we have $F'(0) = 1-\sigma-\frac{a}{4} < 0$, so by (20) the interval I_- lies to the left of 0, and I_+ to the right of 0. Therefore, $I_- \cap I_+ = \emptyset$. Thus, by (20) and (21), there exists an attracting periodic orbit of period 2.

Now we can describe the dynamics of F (and therefore of $f_{ab\sigma}$). Let $P = \{z_-, z_+\}$, where $z_- < 0 < z_+$, be the periodic orbit found there. From the formula for F' it follows that F has two critical points, $\kappa_- < 0$ and $\kappa_+ > 0$. From (20) and (21) it follows that

$$F(\kappa_+) < z_- < \kappa_- < 0 < \kappa_+ < z_+ < F(\kappa_-).$$

In particular, the interval $J = [F(\kappa_+), F(\kappa_-)]$ is invariant. Moreover, the trajectories of both critical points are attracted to P, so by Lemma IV.1, there are no attracting or neutral periodic points except z_- and z_+.

Consider intervals $J_- = [F(\kappa_+), \kappa_-]$ and $J_+ = [\kappa_+, F(\kappa_-)]$. We have $0 \leq F' < 1-\sigma$ on $J_- \cup J_+$, so $F(J_-) \subset J_+$ and $F(J_+) \subset J_-$. Therefore the trajectories of all points from $J_- \cup J_+$ converge to P.

Our map is decreasing on $J_0 = [\kappa_-, \kappa_+]$, and has there a fixed point z_0. Since there are no attracting or neutral periodic points in J_0, trajectories of all points from J_0 (except z_0) are repelled from z_0 and eventually enter $J_- \cup J_+$. Then they are attracted to P. Similar argument shows that trajectories of all points from $\mathbb{R} \setminus J$ eventually enter J, and then they either hit z_0 and stay there, or are attracted by P.

Now Theorem IV.9 follows immediately by the conjugacy argument. □

Theorem IV.9 guarantees that when intensity of choice is large enough, then the system will inevitably converge to an attracting periodic orbit of period 2. Thus, although the system will not converge, the behavior will be predictable. Moreover, the threshold a_1 can be chosen in such a way that for a wide variety of levels of asymmetry of cost functions ($b \in [b_0, 1-b_0]$) if intensity of choice crosses this level each system will be attracted (excluding countably many trajectories falling into equilibrium, but which are almost impossible to get into) to the attracting periodic orbit of period 2. Obviously this attracting periodic orbit has to depend on the values of a, b and σ.

3. *Chaos for* $b \in \left(0, \frac{1-\sigma}{2-\sigma}\right) \cup \left(\frac{1}{2-\sigma}, 1\right)$.

Now we look at the case $b \in \left(0, \frac{1-\sigma}{2-\sigma}\right) \cup \left(\frac{1}{2-\sigma}, 1\right)$. Parameter b is the characteristic of our game – it tells us how different are the costs of resources. Taking b far enough from 1/2 implies that costs of resources are distinguishable for agents who are discounting/forgetting past costs with factor σ. We show that in such case the chaotic behavior emerges.

Theorem IV.11. *For a fixed $\sigma > 0$ and b, if either $b < \frac{1-\sigma}{2-\sigma}$ or $b > \frac{1}{2-\sigma}$, then there exists $a_0 > 0$ such that if $a > a_0$ then $f_{ab\sigma}$ is Li-Yorke chaotic and $h(f_{ab\sigma}) \geq \log\frac{1+\sqrt{5}}{2}$.*

Corollary IV.12. *For fixed $\sigma \in [0,1)$ and $b \in \left(0, \frac{1-\sigma}{2-\sigma}\right) \cup \left(\frac{1}{2-\sigma}, 1\right)$ the system becomes chaotic for sufficiently large values of the intensity of choice.*

In order to prove Theorem IV.11, we will show that if a is sufficiently large then $f_{ab\sigma}$ has a periodic point of period 3.

Proof of Theorem IV.11. First, we will focus on the behavior of critical points of F for large values of a. Put $t = e^{ax}$. Then from (10) the equation for the zeros of F' becomes

$$t^2 + \left(2 - \frac{a}{1-\sigma}\right)t + 1 = 0.$$

If a is sufficiently large, then this equation has two roots, both positive, one less than 1 and the other one larger than 1.

For a given $\varepsilon > 0$, as a goes to infinity, F' converges uniformly to $1-\sigma$ on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$. Thus, if a is sufficiently large, both critical points have to be in $(-\varepsilon, \varepsilon)$.

Assume that $b < \frac{1-\sigma}{2-\sigma}$. We know that F has two critical points $\kappa_- < 0$ and $\kappa_+ > 0$ independent of b. We will show that there is a_0 such that if $a > a_0$ then there is a point $x_0 \in (\kappa_-, \kappa_+)$ such that $F(x_0) = \kappa_-$ and $F^3(x_0) > \kappa_+$.

For fixed σ, critical points converge to 0 as a goes to infinity. Thus, by Lemma IV.8, as a goes to infinity, then $F(\kappa_-)$ goes to $1-b$ and $F(\kappa_+)$ goes to $-b$. We have $F(0) \geq 0 > \kappa_-$ and $F(\kappa_+) < \kappa_-$, so there is a point $x_0 \in (\kappa_-, \kappa_+)$ such that $F(x_0) = \kappa_-$. For a sufficiently large, we get $F^3(x_0)$ arbitrarily close to $(1-\sigma)(1-b) - b$, which is positive, while κ_+ is arbitrarily close to 0. Thus, $F^3(x_0) > \kappa_+$.

Now, we have $F(x_0) < x_0 < F^3(x_0)$, and this implies the existence of a periodic point of period 3 for F (see, e.g.,[3]).

In the case of $b > \frac{1}{2-\sigma}$ we use the first case and the identity $F_{1-b}(-x) = -F_b(x)$.

Existence of a periodic point of period 3 implies Li-Yorke chaos and topological entropy at least $\log\frac{1+\sqrt{5}}{2}$ for F (see, e.g.,[70,3]) and thus for $f_{ab\sigma}$. □

Theorem IV.11 shows us that when the difference in cost functions is substantial if agents choose their strategies with sufficiently large intensity of choice a, then the system will inevitably become chaotic. In such case any long-term behavior will become extremely complex. On the other hand, when the cost functions are similar enough, memory loss (recency bias) makes those costs indistinguishable from the perspective of an agent. In such case when the intensity of choice is large the agents follow an attracting periodic orbit of period 2 (Theorem IV.9). This is a crucial differentiation of the long-term behavior of the system: existence of periodic orbit of period 2 which attracts almost all trajectories implies that although the system does not stabilize, it remains relatively predictable — no matter the initial state of the system, it will converge to period 2 orbit, thus after some time, every even number of iterations of the map will place it close to its previous position. When the system becomes chaotic we land in an unpredictable regime with periodic orbits of different periods, dependence on initial conditions and complicated dynamics.

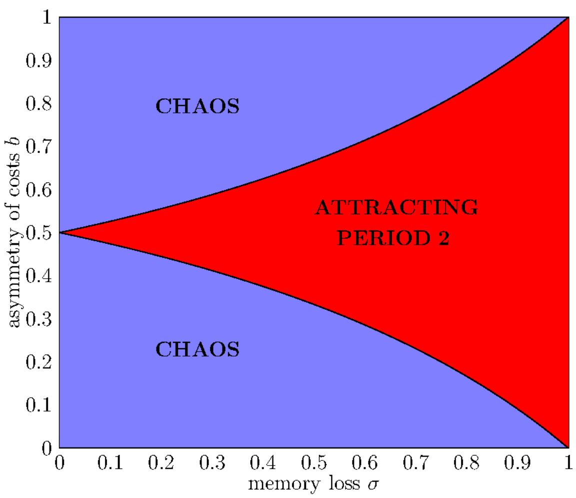


FIG. 5: Behavior of the system for large values of intensity of choice a. As long as $b \in \left(\frac{1-\sigma}{2-\sigma}, \frac{1}{2-\sigma}\right)$, almost all trajectories are attracted to the periodic orbit of period 2. Outside this interval we observe chaotic behavior.

Corollaries IV.10 and IV.12 determine the sets of parameters (σ, b) in which the long-term behavior for large intensity of choice is diametrically different (see Figure 5). When $\sigma = 0$ the interval $\left(\frac{1-\sigma}{2-\sigma}, \frac{1}{2-\sigma}\right)$ shrinks to $\{1/2\}$ and the system will be chaotic if only the cost functions of paths are different. When σ increases the interval where we observe attraction to the orbit of period 2 expands. As σ tends to 1 chaotic behavior vanishes and in the instability region almost all trajectories (except countably many) converge to the attracting periodic orbit of period 2. We finally note that the phase transition at $(\sigma, \frac{1-\sigma}{2-\sigma})$ and $(\sigma, \frac{1}{2-\sigma})$ implies that close to these values a small change of costs of resources (change of b) as well as small change in memory of the agents (change of σ) can push the system from simple periodic behavior to the complex chaotic one (or in the opposite direction).

D. Perturbed costs

In this section we look at the $f_{ab\sigma}$-dynamics through the lenses of MWU algorithm with perturbed (nonlinear) cost functions. We show that if time-average perturbed costs of strategies are convergent, then they converge to the same limit. Lastly, we show that it might happen that they diverge. Throughout this section we assume that $\sigma > 0$. Results for $\sigma = 0$ are discussed in[24].

Recall that the map $f_{ab\sigma}$ that generates our dynamics is defined by

$$f_{ab\sigma}(x) = \frac{x^{1-\sigma}\exp(-\lambda c_1(x))}{x^{1-\sigma}\exp(-\lambda c_1(x)) + (1-x)^{1-\sigma}\exp(-\lambda c_2(1-x))} \tag{22}$$

where $\lambda = \frac{a}{N} = \log\frac{1}{1-\varepsilon}$ and $c_1(x) = \alpha N x$, $c_2(1-x) = \beta N(1-x)$ are costs of the strategies.

The $f_{ab\sigma}$-dynamics can be seen as Multiplicative Weights Update dynamics with perturbed costs functions

$$\begin{aligned} \bar{c}_1(x) &= c_1(x) + \frac{\sigma}{\lambda}\log(x), \\ \bar{c}_2(1-x) &= c_2(1-x) + \frac{\sigma}{\lambda}\log(1-x). \end{aligned} \tag{23}$$

Remark IV.13. *Let $\bar{c}_1, \bar{c}_2$ be defined by (23). Then*

$$f_{ab\sigma}(x) = \frac{x\exp(-\lambda\bar{c}_1(x))}{x\exp(-\lambda\bar{c}_1(x)) + (1-x)\exp(-\lambda\bar{c}_2(1-x))}.$$

We next study properties of time-average perturbed cost of both strategies.

Proposition IV.14.

1. *Let $\{x_0, x_1, \ldots, x_{n-1}\}$ be a periodic orbit of $f_{ab\sigma}$-dynamics of period $n > 0$. Then*

$$\sum_{i=0}^{n-1}\bar{c}_1(x_i) = \sum_{i=0}^{n-1}\bar{c}_2(1-x_i).$$

2. *Let $\{x_0, x_1, x_2, \ldots\}$ be an orbit of $f_{ab\sigma}$-dynamics. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\bar{c}_1(x_i) - \bar{c}_2(1-x_i)) = 0.$$

Moreover, if the sequences $\left(\frac{1}{n}\sum_{i=0}^{n-1}\bar{c}_1(x_i)\right)_{n=1}^{\infty}$ and $\left(\frac{1}{n}\sum_{i=0}^{n-1}\bar{c}_2(1-x_i)\right)_{n=1}^{\infty}$ are convergent, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\bar{c}_1(x_i) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\bar{c}_2(1-x_i). \tag{24}$$

Proof. We first show how perturbed costs can be treated by the conjugate map. We will show that

$$\lambda(\bar{c}_1(x) - \bar{c}_2(1-x)) = a(F(y) - y). \tag{25}$$

By (23) we have

$$\begin{aligned}
&\lambda(\bar{c}_1(x)-\bar{c}_2(1-x)) = \\
&\lambda\left(\alpha N x-\beta N(1-x)-\frac{\sigma}{\lambda}\log\left(\frac{1-x}{x}\right)\right) = \\
&(\alpha+\beta)N\lambda\left(x-\frac{\beta}{\alpha+\beta}\right)-\sigma\log\left(\frac{1-x}{x}\right) = \\
&a(x-b)-\sigma\log\left(\frac{1-x}{x}\right).
\end{aligned} \tag{26}$$

On the other hand

$$\begin{aligned}
&a(F(y)-y) = \\
&a\left[\frac{1}{a}\log\left(\frac{1-f_{ab\sigma}(x)}{f_{ab\sigma}(x)}\right)-\frac{1}{a}\log\left(\frac{1-x}{x}\right)\right] = \\
&\log\left[\left(\frac{1-x}{x}\right)^{1-\sigma}\exp(a(x-b))\right]-\log\left(\frac{1-x}{x}\right) = \\
&a(x-b)-\sigma\log\left(\frac{1-x}{x}\right).
\end{aligned} \tag{27}$$

By (26) and (27) we get

$$\begin{aligned}
\lambda(\bar{c}_1(x)-\bar{c}_2(1-x)) &= a(x-b)-\sigma\log\left(\frac{1-x}{x}\right) \\
&= a(F(y)-y).
\end{aligned}$$

To show the first property it is sufficient to see that by equation (25) we have

$$\begin{aligned}
&\lambda\sum_{i=0}^{n-1}(\bar{c}_1(x_i)-\bar{c}_2(1-x_i)) = a\sum_{i=0}^{n-1}(F(y_i)-y_i) = \\
&a((y_1-y_0)+\ldots+(y_{n-1}-y_{n-2})+(y_0-y_{n-1})) = 0.
\end{aligned}$$

We move to the proof of the second statement. To this aim we will show that

$$I = \left[-\frac{b}{\sigma}, \frac{1-b}{\sigma}\right]$$

is an attracting invariant set. As F_+ and F_- are increasing, $F_+(-\frac{b}{\sigma}) = -\frac{b}{\sigma}$ and $F_-(\frac{1-b}{\sigma}) = \frac{1-b}{\sigma}$ and F is bounded from below by F_+ and from above by F_-, we obtain that $F(I) \subset I$, so I is an invariant, compact set.

Now we will show that I is attracting. We will use the fact that if $x < -\frac{b}{\sigma}$, then $F(x) > x$ and if $x > \frac{1-b}{\sigma}$, then $F(x) < x$.

Take $0 < x < -\frac{b}{\sigma}$. Then $F(x) > x$. So as long as $F^{n+1}(x) \leq -\frac{b}{\sigma}$ we have $F^{n+1}(x) > F^n(x)$. We consider separately when κ_- is inside and outside of I. If $\kappa_- \in I$, then there is no critical point in $(-\infty, -\frac{b}{\sigma})$ and

$$x < F(x) < \cdots < F^n(x) < F^{n+1}(x) \leq -\frac{b}{\sigma}.$$

Thus, there exists n_0 such that $F^{n_0}(x) < -\frac{b}{\sigma}$ and $F^{n_0+1}(x) \geq -\frac{b}{\sigma}$. Since for any $y < -\frac{b}{\sigma}$ we have

$$F(y) \leq F_-\left(-\frac{b}{\sigma}\right) = 1-\frac{b}{\sigma} \leq \frac{1-b}{\sigma},$$

we get that $F^{n_0+1}(x) \in I$.

If the critical point κ is outside of I, that is $\kappa_- < -\frac{b}{\sigma} < 0$. Then, as F has local maximum at κ_- and $-\frac{b}{\sigma} < 0 < \kappa_+$, we get that F is decreasing on $(\kappa_-, -\frac{b}{\sigma})$. Thus, $F(\kappa_-) > F(-\frac{b}{\sigma}) > F_+(-\frac{b}{\sigma}) > -\frac{b}{\sigma}$. Moreover,

$$F(\kappa_-) \leq F_-\left(-\frac{b}{\sigma}\right) = 1-\frac{b}{\sigma} \leq \frac{1-b}{\sigma}.$$

Thus, $F(\kappa_-) \in I$. So trajectory of $x < \kappa_-$ either falls into I (once it crosses the value $F^{-1}(-\frac{b}{\sigma})$) or lands in $(\kappa_-, -\frac{b}{\sigma})$. But then $F((\kappa_-, -\frac{b}{\sigma})) \subset (-\frac{b}{\sigma}, F(\kappa_-)) \subset I$. So trajectory of any $x \in (-\infty, -\frac{b}{\sigma})$ eventually falls into I.

Similar reasoning for $1 > x > \frac{1-b}{\sigma}$, using the fact that $F(x) < x$ when $x > \frac{1-b}{\sigma}$ and $F_+(\frac{1-b}{\sigma}) \geq -\frac{b}{\sigma}$ guarantees that the trajectory of any $x \in (\frac{1-b}{\sigma}, \infty)$ will eventually fall into I.

Now we can show the point 2. By equation (25) we have

$$\lambda\sum_{i=0}^{n-1}(\bar{c}_1(x_i)-\bar{c}_2(1-x_i)) = a\sum_{i=0}^{n-1}(F(y_i)-y_i) = a(y_n-y_0).$$

Because I is a compact attracting invariant set of F-dynamics, there exists $n_0 \in \mathbb{N}$ such that $y_n \in I$ for $n > n_0$. Then for each $n > n_0$

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=0}^{n-1}\left(\bar{c}_1(x_i)-\bar{c}_2(1-x_i)\right)\right| &= \frac{a}{\lambda n}|y_n-y_0| \\
&\leq \frac{a}{\lambda n}\left(\frac{1}{\sigma}+\mathrm{dist}(y_0, I)\right),
\end{aligned}$$

where $\mathrm{dist}(y_0, I) = \inf_{z\in I}|y_0-z|$. Thus,

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=0}^{n-1}\left(\bar{c}_1(x_i)-\bar{c}_2(1-x_i)\right)\right| = 0.$$

□

Corollary IV.15. *Let $\{x_0, x_1, x_2, \ldots\}$ be an orbit of $f_{ab\sigma}$-dynamics. If the sequence $(x_n, 1-x_n)$ is convergent then its limit is an equilibrium of the game. If the sequence $(x_n, 1-x_n)$ is not convergent then any accumulation point of the sequence $(x_n, 1-x_n)$ is an equilibrium of the game.*

Proposition IV.14 guarantees that if time-average perturbed costs of both strategies are convergent, then they are equal to each other. An illustration of the result of Proposition IV.14 is presented in Figure 6. Nevertheless, one may ask if the convergence is guaranteed. The answer is negative. We will show that it may happen that the limits from (24) don't exist.

Assume that the values of parameters are such that f has a periodic orbit of period 2. Let $\varphi\colon [0,1] \mapsto \mathbb{R}$ be a bounded observable, for which the value at the fixed point and the average over the periodic orbit of period

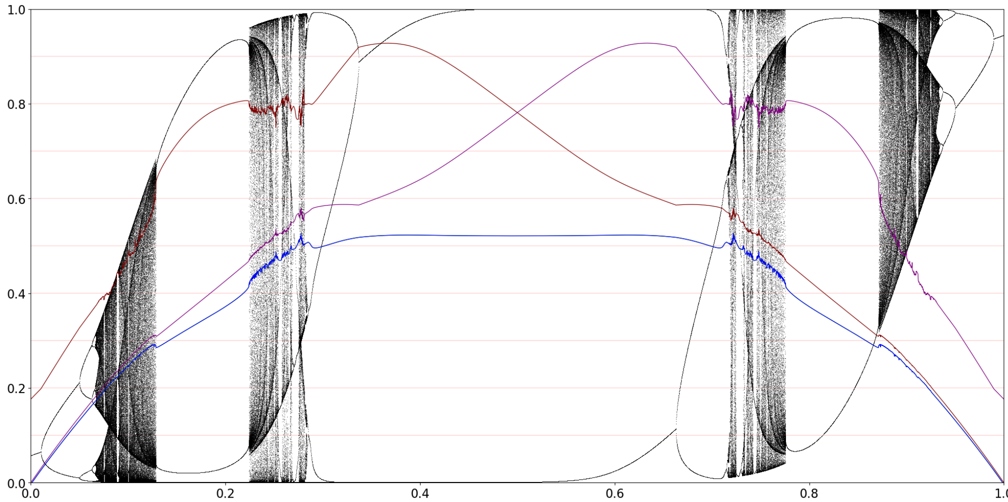


FIG. 6: Time-average costs and time-average perturbed costs for $\sigma = 0.5$, $a = 25$. The horizontal axis represents parameter b for each $b \in [0,1]$, last 300 iterations of the left critical point (κ_l) of the map $f_{ab\sigma}$ out of 20000 iterations are plotted in black. Level lines are in red. Time-average cost of the first (second) strategy is in maroon (purple), time-average perturbed cost of the first (second) strategy is in green (blue). In order to place both iterations and time-average (perturbed) costs on one diagram the costs were scaled down by factor 8. Observe that time-average perturbed cost of the first strategy is invisible as it is covered by time-average perturbed cost of the second strategy.

2 are different − this condition for our game is shown in Figure 7. We want to show that there is a point $x \in (0,1)$ such that the limit of the averages of φ over longer pieces of the orbit of x doesn't exist.

We will use standard tools from the combinatorial dynamics. If I, J are subintervals of $[0,1]$, we say that I *f-covers* J if $J \subset f(I)$. The following lemma can be found for instance in[3].

Lemma IV.16. *Let $(I_i)_{i=0}^{\infty}$ be a sequence of subintervals of $[0,1]$, such that I_i f-covers I_{i+1} for every i. Then there exists a point $x \in I_0$ such that $f_i(x) \in I_i$ for every i.*

We are ready to show that the time average of the observable φ may not converge.

Theorem IV.17. *Assume that f has a unique fixed point $p \in (0,1)$, and a unique period 2 orbit $Q = \{q_1, q_2\}$ (where $q_1 < q_2$). Denote the critical points of f by κ_l, κ_r (where $\kappa_l < \kappa_r$). Assume that*

$$f(\kappa_r) < q_1 < \kappa_l < f^2(\kappa_r) < p < q_2 < \kappa_r < f(\kappa_l), \quad (28)$$

or

$$f(\kappa_r) < q_1 < f^2(\kappa_r) < \kappa_l < p < q_2 < \kappa_r < f(\kappa_l). \quad (29)$$

Under the above assumptions, there exists a point $x \in (0,1)$ such that the sequence of averages $\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x))$ diverges.

Proof. We define five subintervals of $[0,1]$: $A = [f(\kappa_r), q_1]$, $B = [q_1, \kappa_l]$, $C = [\kappa_l, p]$, $D = [p, q_2]$, and $E = [q_2, \kappa_r]$ (see Figure 8).

If p is not repelling, then the trajectory of either C, or $D \cup E$ is attracted to p. However, q_2 belongs to both $f(C)$ and $D \cup E$, so this is impossible. Therefore, p is repelling. Similarly, if Q is not repelling, then the trajectory of one of the three intervals B, $C \cup D$, or E, is attracted to Q. However, p belongs to $f^3(B)$, $C \cup D$, and $f^2(E)$, so this is impossible. Therefore, Q is repelling.

Very similar arguments show that there is no attracting (at least from one side) periodic orbit of period 2, whose one point is in the interior of D and the other point in the interior of C. Similarly, there is no attracting (at least from one side) periodic orbit of period 4, whose one point is in the interior of D and the next point in the interior of B, the next point in the interior of E, and the fourth point in the interior of A.

Now we construct a sequence $(I_i)_{i=0}^{\infty}$ of subintervals of $[0,1]$ in the following way. We start by repeating D, C n_1 times, then we repeat D, B, E, A n_2 times, then D, C n_3 times, then D, B, E, A n_4 times, etc. Let x be the corresponding point, whose trajectory follows this sequence (as in Lemma IV.16). From what we proved in the two preceding paragraphs, it follows, that if the numbers n_i are large enough and grow fast enough, then some pieces of the trajectory of x stay as close to p as we want for as long as we want, and some other pieces of the trajectory of x stay as close to Q as we want for as long as we want. This means that the averages of φ over longer and longer initial pieces of the trajectory of x sometimes approach $\varphi(p)$ as close as we want, and sometimes approach $(\varphi(q_1) + \varphi(q_2))/2$ as close as we want. Since we assumed that those two numbers are different, the limit of the averages of φ does not exist. □

We want to apply this theorem to φ equal to the perturbed cost functions $\bar{c}_1$ and $\bar{c}_2$. One may argue that

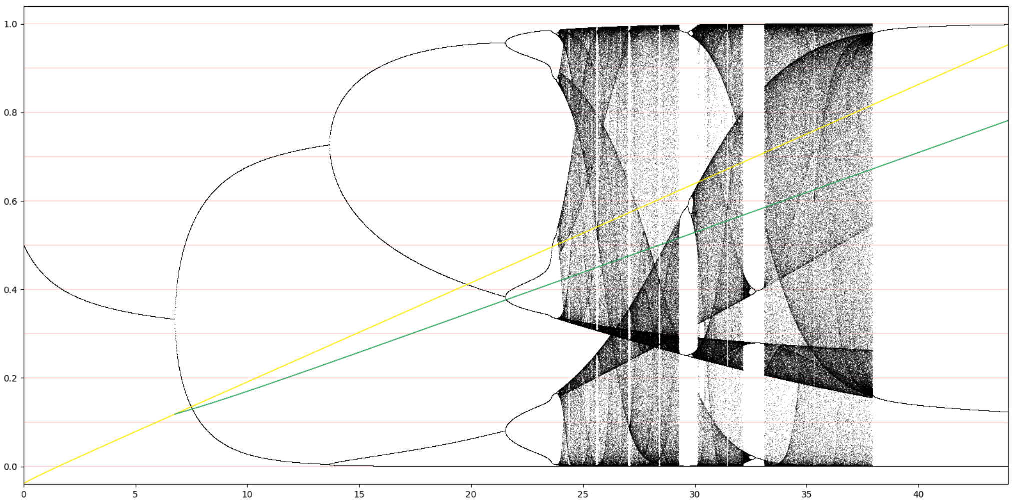


FIG. 7: Bifurcation diagram with perturbed costs of the fixed point $p \in (0,1)$ and periodic orbit $Q = \{q_1, q_2\}$ for $\sigma = 0.5$ and $b = 0.28$, with $a \in [0,44]$. Horizontal axis represents parameter a, for each $a \in [0,44]$ last 200 iterations of the starting point $x_0 = 0.4$ of the map $f_{ab\sigma}$ out of 20000 iterations are plotted in black, time-average perturbed cost of the fixed point p is in yellow, average over the orbit Q is in green. In order to place both iterations and time-average perturbed costs on one diagram the costs were scaled down by factor 9.

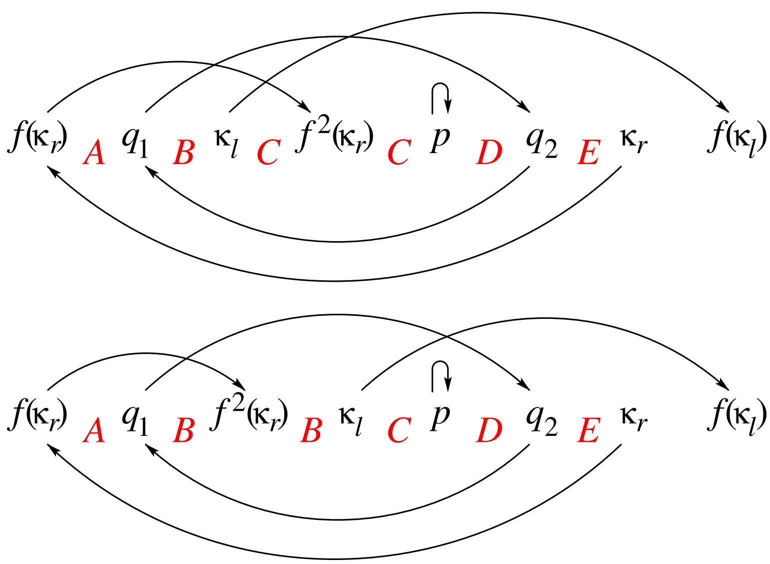


FIG. 8: Dynamics of intervals when inequalities (28), (29) are met.

they are not bounded. However, the core of the map (an invariant interval such that the trajectories of all points except 0 and 1 eventually land in it) is a compact subinterval of $(0,1)$, so those functions restricted to the core are bounded.

It remains to show that for some values of parameters our assumptions are satisfied. We show it numerically in Figure 9. Since the computations do not involve more than two iterates of our map, we do not have to worry about the accumulation of the round-off error.

We see that for our values of σ and b, starting at some value of a the ordering of the important points is as in (28) or (29). It also looks that this should be true for all larger values of a. Note that if $q_2 < \kappa_r$, then we can be sure that $q_1 = f_{ab\sigma}(q_2) > f_{ab\sigma}(\kappa_r)$.

V. CONCLUSIONS

In this paper we show that chaotic behavior can be observed in a large class of EWA dynamics for simple two-strategy nonatomic congestion game. We derive this

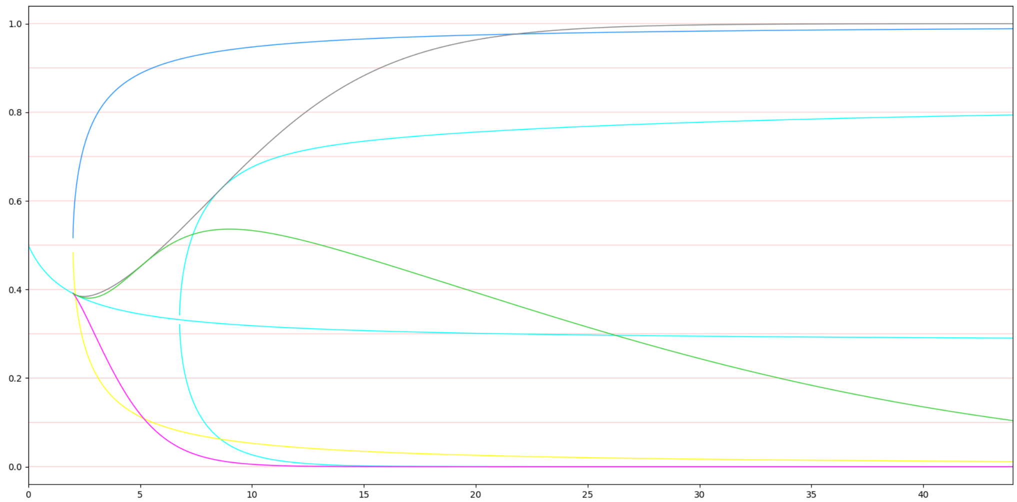


FIG. 9: Points from (28) and (29) for $\sigma = 0.5$ and $b = 0.28$, with $a \in [0, 44]$. Level lines are in red. Periodic points of period 1 and 2 are shown in cyan. Critical point κ_l is in yellow and κ_r in blue; $f_{ab\sigma}(\kappa_r)$ is in gray, $f^2_{ab\sigma}(\kappa_r)$ in green, and $f_{ab\sigma}(\kappa_l)$ in magenta. Observe that the inequalities in (28) are satisfied when values of the parameter a are sufficiently large.

class of dynamics from Galla and Farmer[36]. We show that in such game an increase in the intensity of choice will inevitably result in losing stability of the system. Moreover, the interplay between asymmetry of costs and memory loss will give qualitatively different behaviors for large values of the intensity of choice. For $\sigma = 0$, that is when all previous costs are equally important, the system will become chaotic only if costs of resources are different. When σ increases (memory loss/discount factor increases) the range of values of the parameter of asymmetry of costs b, for which the trajectories of almost all points will be attracted by periodic orbit of period 2, will grow, eventually for $\sigma = 1$ attaining the whole unit interval $(0, 1)$. This behavior gives two completely different regimes. The system where all trajectories are attracted to the periodic orbit of period 2 is predictable and the dynamics is simple, while chaotic regime is unpredictable resulting in complex dynamics.

Our results show that while potential/congestion games are traditionally viewed as one of the most predictable classes of games in terms of their dynamics, their detailed picture is much more complicated. These results are in line with numerous recent findings[9,22,24,25,55,62,67], suggesting that complex and non-equilibrating behavior of agents employing learning rules widely applied in economics seems to be common rather than exceptional.

In addition, we show that memory loss can prevent chaos in two-strategy congestion game with homogeneous population of agents. But what will happen in heterogeneous case? And what if agents have more strategies/resources available? Evidently, the system will be more complicated. Nevertheless, in the full memory case one can observe the emergence of chaotic behavior for $b \neq 1/2$ as a consequence of the increase of the intensity of choice, both in heterogeneous case and for many strategies[24]. We leave the answer to the memory loss case for future work. Moreover, one may ask if these results are algorithm specific. Results on more general classes of dynamics[9,57] suggest that our result can be generalized to larger class of dynamics like (discounted) FTRL dynamics. We also leave the answer to this question for the future work.

Lastly, Pangallo et al.[62] showed that best reply cycles — basic topological structures in games — predict nonconvergence of six well-known learning algorithms that are used in biology or are supported by experiments with human players. Best reply cycles are dominant in complicated and competitive games, indicating that in these cases equilibrium is typically an unrealistic assumption, and one must explicitly model the learning dynamics. These examples of complex and chaotic behavior strongly suggests that chaotic, non-equilibrium results can be further generalized to other games.

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AUTHOR DECLARATIONS

Conflict of interest

The authors have no conflicts to disclose.

Author contributions

All authors contributed equally.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Appendix

In the Appendix we use the notation $\eta = 1 - \sigma$.

Appendix A: Derivation of the dynamics from Experience-Weighted Attraction

In this section we show how to derive the update rule in (2) from Experience-Weighted Attraction (EWA) dynamics. Our approach is based on the derivation of an analogous formula for a game with finite set of players that can be found in Supplementary Information Appendix of[36]. We reproduce this procedure here (with necessary modifications for inclusion of a continuous population) for the convenience of the reader.

We begin with introduction of the basic notation of a game $G = (N, S, \pi)$. The set of agents is $N = \{1, 2, \ldots, n\}$, $n > 1$, the agents are indexed by $i \in N$. We denote $S_i = \{s_i^1, s_i^2, \ldots, s_i^j, \ldots, s_i^m\}$ a strategy space of agent i. For simplicity of notation we assume that all strategy sets have the same size $m > 0$. Then $S = \prod_{i=1}^n S_i$ is the set of all strategy profiles and $S_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ denotes the strategy space of all agents except agent i. Let Δ_i be the set of mixed strategies of agent i, that is a set of all probability distributions over S_i. Then Δ_{-i} is the product set of mixed strategy profiles of all agents except agent i. We denote $\pi_i(x_i, x_{-i})$ a scalar-valued payoff function of agent i when playing strategy $x_i \in \Delta_i$ against strategy profile $x_{-i} \in \Delta_{-i}$ of other agents. Then $\pi = (\pi_1, \ldots, \pi_n)$ is the payoff vector. Let $x_i(t)$ be an actual strategy chosen by agent i in period $t \geq 0$ and let $x_{-i}(t)$ denote actual strategy chosen by other agents in period t. Finally, let $\pi_i(x_i(t), x_{-i}(t))$ be payoff of agent i in period t.

The core of the EWA model is two variables which are updated after each round:

- $N(t)$ which we interpret as the number of 'observation-equivalents' of past experience,
- $Q_i^j(t)$ denotes agent i's attraction of strategy s_i^j after period t has taken place.

The evolution of attractions goes as follows

$$Q_i^j(t+1) = \frac{\eta N(t) Q_i^j(t) + [\delta + (1-\delta) I(s_i^j, s_i(t))] \cdot \pi_i(s_i^j, s_{-i}(t))}{N(t)},$$

where

$$N(t+1) = \eta(1-\kappa) N(t) + 1.$$

The attractions are translated into probabilities by the logit transformation

$$x_i^j(t+1) = \frac{\exp(\lambda Q_i^j(t+1))}{\sum_{k=1}^m \exp(\lambda Q_i^k(t+1))}.$$

The roles of the parameters of EWA are explained below:

- The parameter $\sigma = 1 - \eta \in [0,1]$ describes the rate of discounting of the past experience. For $\sigma = 1$ only the most recent experience affects the agents' decisions, and for $\sigma = 0$ all past experience has equal weight.
- The parameter $\delta \in [0,1]$ is a relative weight given to strategies that are played vs. the strategies that are not played. For $\delta = 1$ agents update all attractions in every step, and for $\delta = 0$ only strategies that are played are being updated.
- The parameter $\kappa \in [0,1]$ affects the way the attractions are aggregated. For $\kappa = 1$ attractions are cumulated, and for $\kappa = 0$ the attractions are taken as an average.
- The parameter $\lambda \geq 0$ is an intensity of choice. For $\lambda = 0$ agents choose strategies with equal probability, and with $\lambda \to \infty$ the agents choose only the strategy with the highest attraction.
- $I(.,.)$ is an indicator function, that is $I(x,y) = 1$ if $x = y$ and $I(x,y) = 0$ otherwise.

In this article we use simplified version of the EWA algorithm. Specifically, we set $\delta = 1$ and $\kappa = 1$, therefore agents update all attractions in every step and attractions are cumulated. Moreover, as we work with congestion games, instead of payoffs we consider costs $c_i(s_i^j, s_{-i})$ resulting from choosing strategy s_i^j against strategy profile of the opponents s_{-i} (where $c_i = -\pi_i$ for every $i \in N$). Then we have

$$Q_i^j(t+1) = \eta Q_i^j(t) - c_i(s_i^j, s_{-i}(t)). \tag{A1}$$

We note that the update rule in (A1) is stochastic, that is agent i will see the realization of the strategy profile of other agents $s_{-i}(t)$ with probability $x_{-i}^{s_{-i}}(t) = \prod_{k \neq i} x_k^{s_k}(t)$. We simplify the problem by considering an adiabatic limit of the process. This procedure corresponds to averaging over batches of a large (infinite) number of rounds between two adaptation steps, that is, to the replacement

$$c_i(s_i^j, s_{-i}(t)) \longrightarrow \tilde{c}_i(s_i^j, x_{-i}(t)) = \sum_{s_{-i}} c_i(s_i^j, s_{-i}(t)) \cdot x_{-i}^{s_{-i}}(t).$$

This simplification leads us to deterministic learning dynamics

$$x_i^j(t+1) = \frac{x_i^j(t)^\eta \exp(-\lambda \tilde{c}_i(s_i^j, x_{-i}(t)))}{\sum_{k=1}^m x_i^k(t)^\eta \exp(-\lambda \tilde{c}_i(s_i^k, x_{-i}(t)))}.$$

And because in our game we have a continuous population and only two strategies, we can drop the indices

$$x(t+1) = \frac{x(t)^\eta \exp(-\lambda c_1(x(t)))}{x(t)^\eta \exp(-\lambda c_1(x(t))) + (1-x(t))^\eta \exp(-\lambda c_2(1-x(t)))}.$$

Thus, we obtain the update rule of the formula (2).

Appendix B: Derivation of the dynamics from multiplicative weights with discounting

We consider a two-strategy non-atomic congestion game where the play is driven by Multiplicative Weight Update (MWU) algorithm with discounting of previous costs. We denote the strategies by s_1, s_2. The cost of each strategy depends on the fraction of agents $x \in [0,1]$ that use strategy s_1

$$c_1(x) = \alpha N x, \qquad c_2(1-x) = \beta N (1-x),$$

where $N > 0$ is the mass of the entire population of agents and $\alpha, \beta \geq 0$, $\max\{\alpha, \beta\} > 0$, are parameters that differentiate the strategies.

At every step $n \geq 0$ the strategies s_1, s_2 have weights $w_1(n)$ and $w_2(n)$ respectively. The initial weights $w_1(0)$ and $w_2(0)$ can be arbitrary positive numbers. Then at step n the strategy s_i is chosen by each agent with probability $\frac{w_i(n)}{w_i(n)+w_j(n)}$, where $i,j \in \{1,2\}$, $i \neq j$. As a result, the fraction of population that uses the strategy s_1 at step $n+1$ is

$$x_{n+1} = \frac{w_1(n+1)}{w_1(n+1)+w_2(n+1)}. \tag{B1}$$

The weights are updated by

$$w_i(n+1) = w_1(0) \cdot (1-\varepsilon)^{\sum_{k=0}^{n} \eta^{n-k} c_1(x_k^i)},$$

where $\varepsilon \in (0,1)$ is a learning rate and $\sigma = 1-\eta \in [0,1]$ is a discount factor that depreciates past costs, with $x_k^1 = x_k$, $x_k^2 = 1 - x_k$. Thus, the weight w_i decreases with higher discounted cumulative cost of previous play of strategy s_i, $i \in \{1,2\}$. We express the update rule of the weights in terms of previous-step weights

$$\begin{aligned} &w_i(n+1) = \\ &w_i(0) \cdot (1-\varepsilon)^{\eta \sum_{k=0}^{n-1} \eta^{n-1-k} c_i(x_k^i)} \cdot (1-\varepsilon)^{c_i(x_n^i)} = \\ &(w_i(n))^\eta \cdot (1-\varepsilon)^{c_i(x_n^i)}. \end{aligned}$$

Then from (B1) we have that

$$\begin{aligned} &x_{n+1} = \\ &\frac{(w_1(n))^\eta \cdot (1-\varepsilon)^{c_1(x_n)}}{(w_1(n))^\eta \cdot (1-\varepsilon)^{c_1(x_n)} + (w_2(n))^\eta \cdot (1-\varepsilon)^{c_2(1-x_n)}}. \end{aligned} \tag{B2}$$

Note that $(1-\varepsilon)^{c_i(\cdot)} = \exp\left[-\log\left(\frac{1}{1-\varepsilon}\right) \cdot c_i(\cdot)\right]$. Therefore, by dividing the numerator and the denominator of (B2) by $(w_1(n)+w_2(n))^\eta \cdot (1-\varepsilon)^{c_1(x_n)}$ we get

$$\begin{aligned} &x_{n+1} = \\ &\frac{x_n^\eta}{x_n^\eta + (1-x_n)^\eta \cdot \exp\left[N \log\left(\frac{1}{1-\varepsilon}\right) \cdot (\alpha x_n - \beta(1-x_n))\right]}. \end{aligned}$$

Then by denoting $a = (\alpha+\beta) N \log\left(\frac{1}{1-\varepsilon}\right)$ and $b = \frac{\beta}{\alpha+\beta}$ we obtain

$$x_{n+1} = \frac{x_n^\eta}{x_n^\eta + (1-x_n)^\eta \exp\left(a\left(x_n - b\right)\right)}.$$